A Novel Layout for Almost Convergent Sequence Spaces

Murat Candan¹, Gülsen Kılınç²

¹Department of Mathematics, İnönü University Malatya-44280, Turkey.
²Department of Mathematics, Adıyaman University Adıyaman-02040, Turkey.

Abstract

The target of the existing study is to acquaint the sequence spaces \( \mathcal{E}(G,F), \mathcal{E}_0(G,F) \) and \( \mathcal{E}_s(G,F) \), where \( G \) is generalized weighted means and \( F \) is the Fibonacci matrix. We describe \( \gamma \) and \( \beta \)-duals of the spaces \( \mathcal{E}(G,F) \) and \( \mathcal{E}_s(G,F) \). Further, we characterize the infinite matrices \( (\mathcal{E}(G,F); \mu) \) and \( (\mu; \mathcal{E}(G,F)) \), where \( \mu \) is an arbitrary sequence space.

Keywords: Almost convergence; Generalized weighted mean; Fibonacci numbers.

1. Introduction

Let us begin with the definition of sequence space, which is the principal notion of summability theory. Let us represent the space of all real-valued sequences by \( w \). A sequence space is identified as any vector subspace of \( w \). By \( l_\alpha, c, c_0, l_\beta \)  \((1 \leq p < \infty)\), bs and cs, we symbolize the sets of all bounded, convergent, null sequences, \( p \) — absolutely convergent series, bounded series and convergent series, seriatim. At the same time, it is going to used emblem that \( e = (1,1,\ldots,1,\ldots) \) and \( e^{(n)} \) is the sequence space whose only non-zero terms is1 in the \( n \)-th place for each \( n \in \mathbb{N} \), where \( \mathbb{N} = \{0,1,2,\ldots\} \). By using optional \( \lambda \) and \( \mu \) sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real numbers \( a_{nk} \), where \( n, k \in \mathbb{N} \), we can defines a matrix transformation as follows: For each \( x = (x_k) \in \lambda \), if \( Ax = (A_n(x)) \), the \( A \) —transform of \( x \), is in \( \mu \); we tell that \( A \) is a matrix transformation from \( \lambda \) into \( \mu \) and we demonstrate such class of all matrices by \( (\lambda, \mu) \). If \( A \) matrix belongs to this class, then the series \( A_n(x) \) is convergence for each \( n \in \mathbb{N} \) and \( x \in \lambda \), where

\[
A_n(x) = \sum_k a_{nk} x_k \quad \text{for each} \quad n \in \mathbb{N},
\]

and \( A_n = (a_{nk})_{k \in \mathbb{N}} \) defines the sequence in the \( n \)-th row of \( A \). To make it easier to write, from now on, the summation without limits ranging from 0 to \( \infty \). A matrix \( A \) is called triangle if primary diagonal’s components aren’t zero and components on the top of the primary diagonal are zero. To tell the truth that \( T(Sx) = (TS)x \) holds for a triangle matrices \( T, S \) and a sequence \( x \). Further, a triangle matrix \( Y \) uniquely has an inverse \( Y^{-1} = Z \) which is also a triangle matrix. Then \( x = Y(Zx) = (YZ)x \) yields for aforementioned matrices and for all \( x \in w \).

When one normed sequence space \( v \) encapsulates a sequence \( (b_n) \) with the feature that for every \( y \in v \), there exists a unexamined sequence of scalars \( (\vartheta_n) \) such that

\[
\lim_{n \to \infty} \|y - (\vartheta_0 b_0 + \vartheta_1 b_1 + \ldots + \vartheta_n b_n)\| = 0
\]

then \( (b_n) \) is said a Schauder basis (or briefly basis) for \( v \). The series \( \sum_k \vartheta_k b_k \) which has the sum \( y \) is called the enlargement of \( y \) according to \( (b_n) \), and written as \( y = \sum_k \vartheta_k b_k \). The notion of schauder basis coincides with algebraic basis for finite sequence spaces.

Now, let’s get to know matrices used in this study. To define first matrix, we need following information: Let \( U \) be the set of all sequences \( u = (u_k) \) such that \( u_k \neq 0 \) for all \( k \in \mathbb{N} \), and for \( u \in U \), let \( \frac{1}{u} = \left( \frac{1}{u_k} \right) \), then, the matrix \( G(u, v) = (g_{nk}) \) which is one of the main matrices in this paper and known generalized weighted mean or factorable matrix is defined following:
\[ g_{nk} = \begin{cases} u_n v_k, & (k < n), \\ u_n v_n, & (k = n), \\ 0, & (k > n), \end{cases} \]

for all \( k, n \in \mathbb{N} \), where \( u, v \in U \), also \( u_n \) depends only on \( n \) and \( v_k \) bounds up with only \( k \).

The other is the Fibonacci matrix defined by the terms of the Fibonacci sequence. Now let us remember the famous Fibonacci sequence used to characterize the Fibonacci matrix. Fibonacci Sequence numbers in such a way that each term of it equals to the sum of the previous terms. In the sequence, the first two terms are 1. If we write it plainly, it is a Fibonacci sequence with the ordinary norm described by

\[ \lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \phi \quad \text{(Golden Ratio)} \]

\[ \sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \text{for each} n \in \mathbb{N}. \]

\[ \sum_{k} \frac{1}{f_k} \text{ converges.} \]

\[ f_{n-1} f_{n+1} - f_n^2 = (-1)^{n+1} \quad \text{for each} n \geq 1 \quad \text{(Cassini Formula)}. \]

Now, let have a look at description of aforenamed matrix. Let \( f_n \) be the \( n \)-th Fibonacci number for each of \( n \in \mathbb{N} \). Then, the Fibonacci matrix \( F = \{ f_{nk} \} \) is described as

\[ f_{nk} = \begin{cases} \frac{f_n}{f_{n+1}}, & (k = n), \\ -\frac{f_n}{f_{n+1}}, & (k = n - 1), \\ 0, & (0 \leq k < n - 1 \text{ or } k > n), \end{cases} \]

for each \( k, n \in \mathbb{N} \).

The domain of an infinite matrix \( K \) on a sequence space \( v \) is sequence space recognized by

\[ v_K = \{ y = (y_k) \in w : Ky \in v \}. \]

Generally, the new sequence space \( v_K \) is the dilation or the becoming smaller of the original space \( v \), in some cases it can be sighted that those spaces overlap. Also, if \( v \) is one of the sequence space of bounded, convergent and null sequence spaces, Then inclusion relationship \( v_K \subset v \) strictly holds.

Combined with a linear topology a sequence space \( \mu \) is said a \( K \)-space, if for each \( i \in \mathbb{N} \), coordinate maps \( p_i: \mu \to \mathbb{C} \), described by \( p_i(x) = x_i \), are continuous, where \( \mathbb{C} \) is the complex numbers field. A \( K \)-space which is a complete linear metric space is said an FK-space. An FK-space whose topology is normable is said a BK-space, which comprises \( \phi \), the set of all finitely nonzero sequences. If \( K \) is taken as a triangular matrix, in that case, we can obviously say that the sequence spaces \( v_K \) and \( v \) are linearly isomorphic, i.e., \( v_K \cong v \) and if \( v \) is a BK-space, then \( v_K \) is also a BK-space with the norm given by \( \| y \|_v = \| Ky \|_v \), for all \( y \in v_K \). As well as above mentioned sequence spaces \( l_{\infty}, c, c_0 \), almost convergent sequence space \( \hat{c} \) are BK-spaces with the usual sup norm defined by

\[ \| y \|_w = \sup_{k \in \mathbb{N}} |y_k|. \]

Also \( l_p \) are BK-spaces with the ordinary norm described by

\[ \| y \|_p = \left( \sum_k |y_k|^p \right)^{1/p}, \quad (1 \leq p < \infty). \]

Let us give very short historical knowledge about the space of almost convergent sequence. This sequence space has two representations \( \hat{c} \) and \( f \). Since the Fibonacci sequence is also denoted by \( f \), in order to avoid a confusion in the representation of this space, we will use the symbol \( \hat{c} \). The almost convergent sequence space was given literature by Lorentz. He used Banach Limits to define that space. The non-negative linear functional \( L \) defined on bounded sequence space \( l_{\infty} \), that satisfies the following conditions is called the Banach Limit. Here are these conditions: \( L(\Psi x) = L(x) \) and \( L(e) = 1 \), where \( \Psi \) is a shift operator defined by \( \Psi(x_n) = x_{n+1} \). A sequence \( y = (y_k) \) is called to be almost convergent to the generalized limit \( l \), if all Banach limits of \( y \) are \( l \), and demonstrated by \( \hat{c} - \lim y_k = l \). In another saying, \( \hat{c} - \lim y_k = l \) iff uniformly in \( n, \lim_{m \to \infty} r_{mn}(y) = l \), where

\[ r_{mn}(y) = \frac{1}{m+1} \sum_{k=0}^{m} y_{k+n}. \]
We demonstrate the sets of all almost null, almost convergent sequences and series by \( \hat{c}_0, \hat{c} \) and \( \hat{c}s \) seriatim and define as follow:

\[
\hat{c} = \{ y = (y_k) \in w : \lim_{m \to \infty} r_{mn}(y) = l, \text{ uniformly in } n \},
\]

\[
\hat{c}s = \{ y = (y_k) \in w : \exists l \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{n} \frac{y_{k+1}}{m+1} = l, \text{ uniformly in } n \}.
\]

It is known that the containment \( c \subset \hat{c} \subset l_{sc} \) are precisely acquired. On account of these containments, norms \( \| \cdot \|_c \) and \( \| \cdot \|_{sc} \) of the spaces \( \hat{c} \) and \( l_{sc} \) are equal. Hence, these sets \( \hat{c} \) and \( \hat{c}_0 \) are BK-spaces getting the following norm

\[
\| y \|_{\hat{c}} = \sup_{m, n} |r_{mn}(y)|.
\]

For a sequence \( y = (y_k) \), we indicate the difference sequence by \( \Delta y = (y_k - y_{k-1}) \). The difference sequence spaces first presented by Kizmaz and defined as follows: \( \lambda(\Delta) = \{ y = (y_k) \in w : \Delta y = (y_k - y_{k+1}) \in \lambda \} \). It was attested by Kizmaz that \( \lambda(\Delta) \) is a Banach space with the norm \( \| y \|_\lambda = |y_1| + \| \Delta y \|_\lambda \). Further, the author examined the \( \alpha -, \beta - \) and \( \gamma - \) duals of the difference spaces and specified the classes \( (\lambda(\Delta); \mu) \) and \( (\mu; \lambda(\Delta)) \) of infinite matrices, where \( \lambda, \mu \in \{ l_{sc}, c \} \).

When we look according to summability theory perspective, we can see that to define new Banach spaces by the matrix domain of triangle and investigate their algebraic, geometrical and topological properties is admitted. Therefore, many authors were interested in this subject and by using some known matrices, many studies were done.

Now, let us look at studies about this topic, recently. The sets \( f(G) \) and \( f_0(G) \) reproduced by the generalized weighted mean have lately been investigated in. Matrix domains of the double band \( B(r,s) \) matrix in the sets of almost null \( f_0 \) and almost convergent \( f \) sequence spaces are studied by Başar and Kıraç. Later, Kayaduman and Şengönül acquire some almost convergent spaces which are being the matrix domains of the Riesz matrix and Cesaro matrix of order 1 in the sets of almost null \( f_0 \) and almost convergent \( f \) sequence spaces. Candan studied on the spaces \( f_0(\tilde{B}) \) and \( f(\tilde{B}) \). Also using generalized difference Fibonacci matrix, Candan and Kayaduman defined \( f^{(r,s)} \) space. Recently, A. Karaçay and F. Özger the spaces \( f(u,v,\Delta), f_0(u,v,\Delta) \) and \( fs(u,v,\Delta) \) and \( f^s(u,v,\Delta) \) defined and studied. Further, it can be seen those works about this topic nearly: [8], [26], [27], [28].

Making use of generalized weighted mean and \( F \) -Fibonacci matrix, we acquaint \( \hat{c}(G,F), \hat{c}_0(G,F) \) and \( \hat{c}s(G,F) \) sequence spaces forming all sequences whose generalized weighted \( F \) - means are in the \( \hat{c}, \hat{c}_0 \) and \( \hat{c}s \) spaces. Now, let’s define the new matrix \( R \) as \( R = R(G,F) = G, F \), where

\[
R(G,F) = \{ r_{nk} \} = \begin{cases} u_n v_k \frac{f_k}{f_{k+1}} - u_n v_{k+1} \frac{f_{k+1}}{f_k}, & k < n, \\ u_n v_k \frac{f_k}{f_{k+1}}, & k = n, \\ 0, & k > n. \end{cases}
\]

Now let us define the sequence space \( \hat{c}(G,F) \),

\[
\hat{c}(G,F) = \{ x = (x_k) \in w : y = (y_k) = (R(G,F)(x) \in \hat{c} \}
\]

where the sequence \( y = (y_k) \) is the \( R(G,F) \) -transform of a sequence \( x = (x_k) \), i.e.,

\[
y_0 = u_0 v_0 \frac{f_0}{f_1} x_0, \quad y_k = \sum_{i=0}^{k-1} u_k \left( v_i \frac{f_i}{f_{i+1}} - v_{i+1} \frac{f_{i+1}}{f_i} \right) x_i + u_k v_k \frac{f_k}{f_{k+1}} x_k,
\]

we shall for brevity

\[
\tilde{y}_{jk} = \frac{f_{j+1}}{f_{j+2}} \left[ \frac{1}{v_j f_j} - \frac{1}{v_{j+1} f_{j+1}} \right],
\]

and for each \( j, k \in \mathbb{N} \) and if \( y = (y_k) = (R(G,F)(x) \in \hat{c} \), it means that \( \exists \alpha \in \mathbb{C} \) such that, uniformly in \( n \),

\[
\lim_{m \to \infty} \sum_{k=0}^{n} u_{n+k} \left[ \sum_{i=0}^{k-1} u_i \left( v_i \frac{f_i}{f_{i+1}} - v_{i+1} \frac{f_{i+1}}{f_i} \right) x_i + v_{n+k} \frac{f_{n+k}}{f_{n+k+1}} \right] = l.
\]

Similarly, we may define \( \hat{c}_0(G,F) \) and \( \hat{c}s(G,F) \) spaces as

\[
\hat{c}_0(G,F) = \{ x = (x_k) \in w : y = (y_k) = (R(G,F)(x) \in \hat{c}_0 \}
\]

if \( y = (y_k) \in \hat{c}_0 \), we know that in (4), \( \alpha = 0 \). Further,

\[
\hat{c}s(G,F) = \{ x = (x_k) \in w : y = (y_k) = (R(G,F)(x) \in \hat{c}s \}
\]

i.e. \( y = (y_k) = (R(G,F)(x) \in \hat{c}s \), then \( \exists l \in \mathbb{C} \) uniformly in \( n \).
\[
\lim_{m \to \infty} \frac{1}{m+1} \sum_{k=0}^{m} \left( \sum_{j=0}^{k+n} u_j \left( \sum_{i=0}^{j-1} \left( \frac{f_i}{f_{i+1}} - \frac{f_{i+2}}{f_{i+1}} \right) x_i + v_j \right) \right) = l.
\]

We can redefine the spaces \( \breve{c}(G,F), \breve{c}(G,F) \) and \( c_0(G,F) \) by the notation of (1)
\[
\breve{c}_0(G,F) = (\breve{c}_0 \circ R(G,F)), \quad \breve{c}(G,F) = (\breve{c} \circ R(G,F)) \quad \text{and} \quad \breve{c}\hat{s}(G,F) = (\breve{c}\hat{s} \circ R(G,F)).
\]

In this paper, we investigate some topological properties, beta- and gamma- duals of those spaces and study to obtain some of the matrix characterizations between these spaces and standard spaces.

2. Some Algebraic and Topological Characteristics of those Spaces

**Theorem 1** i) The sequence space \( \breve{c}(G,F) \) is normed space with

\[
||x||_{\breve{c}(G,F)} = \sup_{m,n} \left| \frac{1}{m+1} \sum_{k=0}^{m} \left( \sum_{j=0}^{k+n} u_0 + \frac{1}{f_k} \sum_{j=0}^{k+n} v_j \right) \right|,
\]

ii) The sequence space \( \breve{c}\hat{s}(G,F) \) is normed space with

\[
||x||_{\breve{c}\hat{s}(G,F)} = \sup_{m,n} \left| \frac{1}{m+1} \sum_{k=0}^{m} \left( \sum_{j=0}^{k+n} u_0 + \frac{1}{f_k} \sum_{j=0}^{k+n} v_j \right) \right|.
\]

**Theorem 2** The sets \( \breve{c}(G,F), \breve{c}_0(G,F) \) and \( \breve{c}\hat{s}(G,F) \) are linearly isomorphic to the sets \( \breve{c}, \breve{c}_0 \) and \( \breve{c}\hat{s} \) respectively, i.e.,

\( \breve{c}(G,F) \cong \breve{c}, \breve{c}_0(G,F) \cong \breve{c}_0, \breve{c}\hat{s}(G,F) \cong \breve{c}\hat{s} \).

**Proof.** Firstly, let us prove that \( \breve{c}(G,F) \cong \breve{c} \). For this aim, we have to show that a linear bijection does exist between the spaces \( \breve{c}(G,F) \) and \( \breve{c} \). Let us take into consideration the transformation \( T \) described by the notation of (1) from \( \breve{c}(G,F) \) to \( \breve{c} \) by \( x \rightarrow y = T x = R(G,F) x \in \breve{c} \), for \( x \in \breve{c}(G,F) \). It is obvious that \( T \) is linear. Additionally, it is clear that \( x = 0 \) whenever \( T x = 0 \) and on account of this \( T \) is injective.

Let \( y = (y_k) \in \breve{c} \) and define \( x = (x_k) \) by

\[
x_k = \sum_{j=0}^{k-1} \frac{1}{u_j} \bar{v}_{kj} y_j + \frac{1}{f_k} \sum_{j=0}^{k} v_k y_j,
\]

Then we get for \( k \geq 1 \)

\[
u_k \left[ \sum_{j=0}^{k-1} \left( \frac{v_j}{f_{j+1}} - \frac{v_{j+1} f_{j+2}}{f_{j+1}} \right) \left( \sum_{i=0}^{j-1} \frac{1}{u_i} \bar{v}_{ji} y_i + \frac{1}{f_j} \sum_{i=0}^{j} u_i \bar{v}_{ij} y_i \right) \right] + u_k v_k = y_k
\]

for all \( k \in \mathbb{N} \), which takes us to the fact that

\[
\lim_{m \to \infty} \frac{1}{m+1} \sum_{k=0}^{m} y_{k+n} - \lim_{m \to \infty} \frac{1}{m+1} \sum_{k=0}^{m} u_{n+k} \left[ \sum_{i=0}^{n+k-1} \left( \frac{v_i}{f_{i+1}} - \frac{v_{i+1} f_{i+2}}{f_{i+1}} \right) x_i + u_{n+k} \frac{f_{n+k}}{f_{n+k+1}} x_{n+k} \right] = \breve{c} - \lim y_k.
\]

It means that \( x = (x_k) \in \breve{c}(G,F) \). On account of this, we attain the truth that \( T \) is surjective. So, \( T \) is a linear bijection and it means that the spaces \( \breve{c}(G,F) \) and \( \breve{c} \) are linearly isomorphic, as was wished. The fact \( \breve{c}_0(G,F) \cong \breve{c}_0 \) can be analogously verified.

Since, known that the matrix domain \( \lambda_\alpha \) of a normed sequence space \( \lambda \) having a basis necessary and sufficient condition \( \lambda \) has a basis whenever \( A = (a_{nk}) \) is a triangle (Remark 2.4) and since the space \( \breve{c} \) has no Schauder basis, we have;

**Corollary 1** The space \( \breve{c}(G,F) \) has no Schauder basis.

3. The \( \alpha - \beta - \gamma \) -Duals of these Spaces

The \( \alpha - \beta - \gamma \) -duals of the sequence space \( X \) are described by

\[
X_\alpha = \{ a = (a_k) \in w : ax = (a_k x_k) \in l_1 \text{ for all } x = (x_k) \in X \},
\]

\[
X_\beta = \{ a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X \},
\]

and
Theorem 5 The $\gamma$ -dual of the space $\hat{\mathcal{E}}(G,F)$ is the intersection of the sets

$$\mathcal{E}(G,F)^\gamma = b_6 \cap b_7,$$

where $b_6 = \{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k |a_{nk}| < \infty \}$ and $b_7 = \{ a = (a_k) \in w : \lim_{k \to \infty} e_{nk} = 0 \}$. The conclusion may be obtained analogously as told in the proof of Theorem 3 with Lemma 1 in place of Lemma 4(iii). Thus, we disregard details.

Theorem 6 Defined the sets
\[
\begin{align*}
    b_B = \{a = (a_k) \in \mathbb{w} : \lim_{n \to \infty} \sum_{k=n}^{\infty} |\Delta^2 e_{nk}| < \infty\}.
\end{align*}
\]

Then \(\mathcal{S}(G, F)^\beta = b_3 \cap b_6 \cap b_7 \cap b_8\). This can be achieved with a similar concept mentioned in the proof of theory.

**Proof.** This can be acquired with a similar concept talked about in the proof of Theorem 4 with Lemma 2 in place of Lemma 4(iv). So, we disregard details.

### 4. Some Matrix Transformation

We shall use representations following throughout all over the section for brevity that

\[
a_{nk} = \sum_{j=0}^{m} a_{jk} a(n, k, m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j, k} \Delta a_{nk} = a_{nk} - a_{n, k+1}.
\]

**Theorem 7** Let \(\lambda\) be an FK-space, \(U\) be a triangle, \(V\) be its inverse and \(\mu\) be optional subset of \(w\). Thus we obtain \(A = (a_{nk}) \in (\lambda_{U};\mu)\) necessary and sufficient condition

\[
C^{(n)} = (c_{nk}) \in (\lambda, c) f o r a l l n \in \mathbb{N},
\]

\[
C = (c_{nk}) \in (\lambda; \mu),
\]

where,

\[
c^{(n)}_{nk} = \begin{cases} 
    \sum_{j=k}^{m} a_{nj} v_{jk}, & 0 \leq k \leq m, \\
    0, & k > m,
\end{cases} \text{and } c_{nk} = \sum_{j=k}^{\infty} a_{nj} v_{jk} \text{ for all } k, m, n \in \mathbb{N}.
\]

**Lemma 3** \(A \in (\hat{\varepsilon}; \hat{\varepsilon})\) necessary and sufficient condition

\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |a_{nk}| < \infty,
\]

\[
\hat{\varepsilon} - \lim_{n \to \infty} a_{nk} = \alpha_k, \text{ exist for each fixed } k \in \mathbb{N}
\]

\[
\hat{\varepsilon} - \lim_{m \to \infty} \sum_{k=0}^{m} a_{nk} = \alpha,
\]

\[
\lim_{m \to \infty} \sum_{k=0}^{m} |\Delta a(n, k, m) - \alpha_k| = 0 \text{ uniformly in } m.
\]

For an infinite matrix \(A = (a_{nk})\), we shall use representation following for briefness that,

\[
da_{nk}^n = \bar{a}_{nk}(m) = \frac{f_{nk+1}}{f_k u_k v_k} a_{nk} + \frac{1}{u_k} \sum_{j=n+1}^{m} \overline{v}_{jk} a_{nj},
\]

and

\[
da_{nk} = \bar{a}_{nk} = \frac{f_{nk+1}}{f_k u_k v_k} a_{nk} + \frac{1}{u_k} \sum_{j=k+1}^{m} \overline{v}_{jk} a_{nj},
\]

\[
\bar{a}_{nk} = u_n \left( \sum_{i=0}^{n-1} v_i f_{i+1} - v_{i+1} f_{i+1} \right) a_{ik} + \frac{f_k}{f_{n+1}} v_n a_{nk}
\]

for all \(n, k, m \in \mathbb{N}\).

**Theorem 8** Assume that the entries of the infinite matrices given by \(A = (a_{nk})\) and \(H = (h_{nk})\) are linked to the following connection

\[
h_{nk} = \bar{a}_{nk}
\]

for all \(k, n \in \mathbb{N}\) and \(\mu\) be an optional sequence space. Then, \(A \in (\hat{\varepsilon}(G, F); \mu)\) iff \(\{a_{nk}\}_{k \in \mathbb{N}} \in \hat{\varepsilon}(G, F)^\beta\) for all \(n \in \mathbb{N}\) and \(H \in (\hat{\varepsilon}, \mu)\).

**Proof.** Let us assume that \(\mu\) be an arbitrary sequence space and presume that ((12)) retains between the entries of and take \(A = (a_{nk})\) and \(H = (h_{nk})\), also, take notice that the spaces \(\hat{\varepsilon}(G, F)\) and \(\hat{\varepsilon}\) are linearly isomorphic.

We take \(A \in (\hat{\varepsilon}(G, F); \mu)\) and any \(y = (y_k) \in \hat{\varepsilon}\). Thus, \(H, R(G, F)\) does exist and \(\{a_{nk}\}_{k \in \mathbb{N}} \in n_{k=1}^{\beta} b_k\) which supplies that \(\{h_{nk}\}_{k \in \mathbb{N}} \in I_1\) for each \(n \in \mathbb{N}\). Hence, \(H y\) exists and in this way

\[
\sum_{k} h_{nk} y_k = \sum_{k} a_{nk} x_k \text{ for all } n \in \mathbb{N}
\]

we obtain using by ((12)) that \(H y = Ax\), which carries us to outcome \(H \in (\hat{\varepsilon}: \mu)\).

Inversely, let \(\{a_{nk}\}_{k \in \mathbb{N}} \in C^\beta\) for each \(n \in \mathbb{N}\) and \(F \in (\hat{\varepsilon}; \mu)\) yield, and get any \(x = (x_k) \in \hat{\varepsilon}(G, F)\). Then, \(Ax\) exists. That’s why, we acquire from the equality, for each \(n \in \mathbb{N}\)
as $m \to \infty$ that $Ax = Hy$ and it means that $A \in \mathcal{C}(G,F; \mu)$. So, the proof is completed.

Theorem 9 $A \in \mathcal{C}(G,F; c)$ necessary and sufficient condition $D^{(n)} = (d_{mk}^{(n)}) \in (c; c)$ and $D = (d_{mk}) \in (\hat{c}; c)$.

Theorem 10 $A \in \mathcal{C}(G,F; l_u)$ necessary and sufficient condition $D^{(n)} = (d_{mk}^{(n)}) \in (\hat{c}; c)$ and $D = (d_{mk}) \in (\hat{c}; l_u)$.

If we change the roles of the spaces $\mathcal{C}(G,F)$ and $\hat{c}$ with $\mu$, we have;

Theorem 11 Assume that the entries of the infinite matrices $A = (a_{nk})$ and $L = (l_{nk})$ are linked to the connection $l_{nk} = \hat{a}_{nk}, (11)$, for all $k, n \in \mathbb{N}$ and $\mu$ be a previously given sequence space. Then, $A \in (\mu; \mathcal{C}(G,F))$ necessary and sufficient condition $L x \in \hat{c}$.

Proof. Let $x = (x_k) \in \mu$ and take into consideration the subsequent equality

$$
\{R(G,F)(Ax)\}_n = u_n \left( \sum_{j=0}^{n-1} \left( f_j f_{j+1} v_j f_{j+2} v_{j+1} \right) (Ax)_j + f_{n+1} u_n v_n (Ax)_n \right)
$$

$$
= u_n \left( \sum_{j=0}^{n-1} \left( f_j f_{j+1} v_j f_{j+2} v_{j+1} \right) \sum_j a_{nj} x_j + f_{n+1} u_n v_n \sum_k a_{nk} x_k \right)
$$

$$
= \sum_k \left( \sum_{j=0}^{n-1} u_k \left( f_j f_{j+1} v_j f_{j+2} v_{j+1} \right) a_{nj} x_j + f_{n+1} u_n v_n a_{nk} x_k \right)
$$

which results $Ax \in \mathcal{C}(G,F)$ necessary and sufficient condition $L x \in \hat{c}$.

So, the proof is completed.

It is time to present the conditions which are going to be used in the proofs:

$$
\sup_{n \in \mathbb{N}} \left| \sum_k a_{nk} \right| < \infty,
$$

(15)

$$
\lim_{m \to \infty} a_{nk} = a_k, \text{ for each fixed } k \in \mathbb{N},
$$

(16)

$$
\lim_{m \to \infty} \sum_k a_{nk} = a,
$$

(17)

$$
\lim_{m \to \infty} \sum_k |A(a_{nk} - a_k)| = 0,
$$

(18)

$$
\sup_{n \in \mathbb{N}} \left| \sum_k A(a_{nk}) \right| < \infty,
$$

(19)

$$
\lim_{k \to \infty} a_{nk} = 0, \text{ for each fixed } n \in \mathbb{N},
$$

(20)

$$
\lim_{n \to \infty} \sum_k |A^2 a_{nk}| = a,
$$

(21)

$$
\hat{c} - \lim_{m \to \infty} a_{nk} = a_k, \text{ exists, for each fixed } k \in \mathbb{N},
$$

(22)

$$
\lim_{m \to \infty} \sum_k |a(n, k, m) - a_k| = 0, \text{ uniformly in } n,
$$

(23)

$$
\hat{c} - \lim_{m \to \infty} a_{nk} = a,
$$

(24)

$$
\lim_{m \to \infty} \sum_k |A(a(n, k, m) - a_k)| = 0, \text{ uniformly in } n
$$

(25)

$$
\lim_{q \to \infty} \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} |\Delta a(n+i, k) - a_k| \right| = 0, \text{ uniformly in } n,
$$

(26)

$$
\sup_{n \in \mathbb{N}} \sum_k |\Delta a(n, k)| < \infty,
$$

(27)

$$
\hat{c} - \lim_{m \to \infty} (a(n,k)) = a_k, \text{ exists, for each fixed } k \in \mathbb{N}
$$

(28)

$$
\lim_{q \to \infty} \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} |A^2 [a(n+i, k) - a_k]| \right| = 0, \text{ uniformly in } n,
$$

(29)

$$
\sup_{n \in \mathbb{N}} \sum_k |a(n, k)| < \infty,
$$

(30)

$$
\sum_n a_{nk} = a_k, \text{ for each fixed } k \in \mathbb{N},
$$

(31)

$$
\sum_n \sum_k a_{nk} = a,
$$

(32)
Let $A = (a_{nk})$ be an infinite matrix. Thus, the subsequent expressions retain:

**Lemma 4**  

i) $A = (a_{nk}) \in (\ell; \ell')$ necessary and sufficient condition (15), (22) and (23) retain.

ii) $A = (a_{nk}) \in (c; \ell')$ necessary and sufficient condition (15), (22), (24) and (25) retain.

iii) $A = (a_{nk}) \in (\ell; l_{\infty})$ necessary and sufficient condition (19) and (20) retain.

iv) $A = (a_{nk}) \in (\ell'; c)$ necessary and sufficient condition (16), (19) and (21) retain.

v) $A = (a_{nk}) \in (c; \ell')$ necessary and sufficient condition (15), (22) and (24).

vi) $A = (a_{nk}) \in (b s: \ell')$ necessary and sufficient condition (19), (20), (22) and (26).

vii) $A = (a_{nk}) \in (\ell'; c)$ necessary and sufficient condition (20), (22), (25) and (26) retain.

viii) $A = (a_{nk}) \in (c s: \ell')$ necessary and sufficient condition (19) and (22) retain.

ix) $A = (a_{nk}) \in (b s; \ell s)$ necessary and sufficient condition (20), (26) and (28) retain.

x) $A = (a_{nk}) \in (c s: \ell s)$ necessary and sufficient condition (26) and (29) retain.

xi) $A = (a_{nk}) \in (c s: \ell s)$ necessary and sufficient condition (27) and (28) retain.

xii) $A = (a_{nk}) \in (\ell; c s)$ necessary and sufficient condition (30) and (33) retain.

**Corollary 2** The following expressions retain:

i) $A = (a_{nk}) \in (\ell (G, F); l_{\infty})$ necessary and sufficient condition $\{a_{nk}\}_{k \in \mathbb{N}} \in \ell (G, F)^{\beta}$ for all $n \in \mathbb{N}$ and (15) retains with $\hat{a}_{nk}$ in place of $a_{nk}$.

ii) $A = (a_{nk}) \in (\ell (G, F); c)$ necessary and sufficient condition $\{a_{nk}\}_{k \in \mathbb{N}} \in \ell (G, F)^{\beta}$ for all $n \in \mathbb{N}$ and (15), (16), (18) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

iii) $A = (a_{nk}) \in (c (G, F); b s)$ necessary and sufficient condition $\{a_{nk}\}_{k \in \mathbb{N}} \in \ell (G, F)^{\beta}$ for all $n \in \mathbb{N}$ and (30) retains.

iv) $A = (a_{nk}) \in (\ell (G, F); c s)$ necessary and sufficient condition $\{a_{nk}\}_{k \in \mathbb{N}} \in \ell (G, F)^{\beta}$ for all $n \in \mathbb{N}$ and (30), (33) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

**Corollary 3** The following expressions retain:

i) $A = (a_{nk}) \in (l_{\infty}; \ell (G, F))$ necessary and sufficient condition (15), (22) and (23) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

ii) $A = (a_{nk}) \in (\ell; \ell (G, F))$ necessary and sufficient condition (15), (22), (24) and (25) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

iii) $A = (a_{nk}) \in (c; \ell (G, F))$ necessary and sufficient condition (15), (22) and (24) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

**Corollary 4** The following expressions retain:

i) $A = (a_{nk}) \in (b s; \ell (G, F))$ necessary and sufficient condition (19), (20), (22) and (26) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

ii) $A = (a_{nk}) \in (\ell s; \ell (G, F))$ necessary and sufficient condition (20), (22) and (26) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

iii) $A = (a_{nk}) \in (c s; \ell (G, F))$ necessary and sufficient condition (19), (22) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

**Corollary 5** The following expressions retain:

i) $A = (a_{nk}) \in (b s; \ell s (G, F))$ necessary and sufficient condition (20), (26) and (28) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

ii) $A = (a_{nk}) \in (\ell s; \ell s (G, F))$ necessary and sufficient condition (26) and (29) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

iii) $A = (a_{nk}) \in (c s; \ell s (G, F))$ necessary and sufficient condition (27) and (28) retain with $\hat{a}_{nk}$ in place of $a_{nk}$.

5. **Note**

This article is the written version of the authors’ plenary talk delivered on May 11-13, 2017 at International Conference on Mathematics and Mathematics Education ICMME in Şanlıurfa, Turkey.

**References**