Abstract

Non-negative sequences $x$ and $y$ satisfying $\lim_{k \to \infty} M \left( \frac{|x_k - y_k|}{\rho} \right) = 0$ for some $\rho > 0$ are called $M$- asymptotically equivalent of multiple $L$, where $x = (x_k)$ and $y = (y_k)$. Similarly, the strong $M$-asymptotically equivalence is obtained for $L = 1$ by using an Orlicz function $M$. This study contains some new definitions and related theorems about asymptotically equivalent sequences by using a lacunary sequence $\theta = (k_r)$, a strictly positive sequence $p = (p_k)$ and an Orlicz function.

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1. Preliminaries, Notations and Introduction

The space of all real (or complex) valued sequences is denoted by $\omega$. Any vector subspaces of $\omega$ is called a sequence space. A sequence $\theta = (k_r)$ of positive integers is called lacunary if $0 < \rho < \infty$ and $\frac{h_r}{h_{r-1}} \to \infty$ as $r \to \infty$, where $h_r = k_r - k_{r-1}$. We get the intervals $I_r = (k_{r-1}, k_r]$ via the sequence $\theta$ and we denote the ratio $k_r/k_{r-1}$ by $q_r$. The space $N_\theta$ of lacunary strongly convergent sequences was defined by Freedman et al. [4], $N_\theta = \{x = (x_i) \in \omega : \lim_{r \to \infty} \frac{1}{p_r} \sum_{i \in I_r} |x_i - s| = 0$ for some $s\}$. Orlicz [8] used the idea of Orlicz function to construct the space $L^M$. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function $M$ is said to satisfy the $\Delta_2 -$ condition for all values of $u$ if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)(u \geq 0)$. It is easy to see that $K > 2$. The $\Delta_2 -$ condition is equivalent to the inequality $M(Lu) \leq KLM(u)$ for all values of $u$ and $L > 1$. The inequality $M(\lambda x) < \lambda M(x)$ is provided for all $\lambda, \lambda \in (0,1)$ by an Orlicz function.

We shall give the well known inequality below which will be used throughout the paper;

$$|w_i + z_i|^{p_i} \leq T(|w_i|^{p_i} + |z_i|^{p_i})$$

where $w_i$ and $z_i$ are complex numbers, $T = \max(1, 2^{H-1})$ and $H = \text{supp}_1 < \infty$.

Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [5]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices in [7].

2. Main Results

We shall begin with the following definitions. For simplicity, here and in what follows, the function $M$ is any Orlicz function.

Definition 2.1 Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences and $M$ is given any Orlicz function. If the equality $\lim_{k \to \infty} M \left( \frac{|x_k - y_k|^{p_k}}{p_k} \right) = 0$ holds for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are $M$-asymptotically...
Theorem 2.8 which is together with the inequality $| \sum_{k=1}^{n} x_k - y_k | \leq \sum_{k=1}^{n} | x_k - y_k |$, we use the fact that $\sum_{k=1}^{\infty} | x_k - y_k | < \infty$, holds for every $\varepsilon > 0$ and for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are $M$-asymptotically lacunary statistical equivalent of multiple $L$, and is denoted by $x \overset{ML}{\sim} y$. If $L = 1$ then it is said that the sequences $x$ and $y$ are $M$-asymptotically lacunary statistical equivalent.

Definition 2.2 Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences and $M$ is given any Orlicz function. If $\frac{1}{n} \left\{ \sum_{k=n}^{\infty} M \left( \frac{|x_k - y_k|}{\rho} \right) \right\} \rightarrow 0$, as $n \rightarrow \infty$, $\forall \varepsilon > 0$ and for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are $M$-asymptotically statistical equivalent, that is, $x \overset{MS}{\sim} y$. If $L = 1$, we say $x$ and $y$ are $M$-asymptotically statistical equivalent.

Definition 2.3 Let $M$ be any Orlicz function, $\theta$ be a lacunary sequence and $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences. If the equality $\lim_{r \rightarrow \infty} \frac{1}{n_r} \left\{ \sum_{k \in I_r} M \left( \frac{|x_k - y_k|}{\rho} \right) \right\} \rightarrow 0$, holds for every $\varepsilon > 0$ and for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are $M$-asymptotically lacunary statistical equivalent of multiple $L$, and is denoted by $x \overset{M^{L}}{\sim} y$. If $L = 1$ then it is said that the sequences $x$ and $y$ are $M$-asymptotically lacunary statistical equivalent.

Definition 2.4 Let $M$ be any Orlicz function and $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences. If the equality $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} M \left( \frac{|x_k - y_k|}{\rho} \right) = 0$ holds for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are strong $M$-asymptotically lacunary statistical equivalent of multiple $L$, and is denoted by $x \overset{M^{L}}{\sim} y$. If $L = 1$ then it is said that the sequences $x$ and $y$ are strong $M$-asymptotically lacunary statistical equivalent.

Definition 2.5 Let $M$ be any Orlicz function, $\theta$ be a lacunary sequence and $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences. If the equality $\lim_{r \rightarrow \infty} \frac{1}{n_r} \sum_{k \in I_r} M \left( \frac{|x_k - y_k|}{\rho} \right) = 0$ holds for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are strong $M$-asymptotically lacunary equivalent of multiple $L$, and is denoted by $x \overset{M^{L}}{\sim} y$. If $L = 1$ then it is said that the sequences $x$ and $y$ are strong $M$-asymptotically lacunary equivalent.

Definition 2.6 Let $M$ be any Orlicz function, $\theta$ be a lacunary sequence, $(p_k)$ be a bounded sequence of positive real numbers and $x = (x_k)$ and $y = (y_k)$ be the sequences of non-negative real numbers. If $\frac{1}{n_r} \sum_{k \in I_r} M \left( \frac{|x_k - y_k|}{\rho} \right) \rightarrow 0$, as $r \rightarrow \infty$, for some $\rho > 0$, then it is said that the sequences $x$ and $y$ are strong $M(p)^{-\theta}$ asymptotically lacunary equivalent of multiple $L$, and is denoted by $x \overset{M(p)^{-\theta}}{\sim} y$. If $L=1$ then it is said that the sequences $x$ and $y$ are strong $M(p)^{-\theta}$ asymptotically lacunary equivalent.

Theorem 2.7 If $M$ is an Orlicz function satisfy $\Delta_2$-condition, then strong asymptotically equivalence implies strong $M$-asymptotically equivalence, that is, $(x \overset{W}{\sim} y)$ implies $(x \overset{MW}{\sim} y)$. 

Proof Suppose that $x \overset{W}{\sim} y$. For $\varepsilon > 0$, let us choose $0 < \delta < 1$ such that $M(u) < \varepsilon$ for every $u$ with $0 \leq u \leq \delta$. So, the equality $\frac{1}{n} \sum_{k=1}^{n} M \left( \frac{|x_k - y_k|}{\rho} \right) < \varepsilon$ holds, where the first summation is over $\frac{|x_k - y_k|}{\rho} \leq \delta$ and the second summation over $\frac{|x_k - y_k|}{\rho} > \delta$. Since $M$ is continuous, $\frac{1}{n} \sum_{k=1}^{n} M \left( \frac{|x_k - y_k|}{\rho} \right) < \varepsilon$ and for $\frac{|x_k - y_k|}{\rho} > \delta$ we use the fact that $M \left( \frac{|x_k - y_k|}{\rho} \right) < 1 + \frac{|x_k - y_k|}{\rho}$. Since $M$ is non-decreasing and convex, $M \left( \frac{|x_k - y_k|}{\rho} \right) < 1 + \frac{|x_k - y_k|}{\rho}$. Therefore, $M \left( \frac{|x_k - y_k|}{\rho} \right) < \frac{1}{2} K \left( \frac{|x_k - y_k|}{\rho} \right)$. Since $M$ satisfies the $\Delta_2$-condition. Hence, we get the inequality $\frac{1}{n} \sum_{k=1}^{n} M \left( \frac{|x_k - y_k|}{\rho} \right) \leq \left( \frac{K M(2)}{\delta} \right) \frac{1}{n} \sum_{k=1}^{n} \left( \frac{|x_k - y_k|}{\rho} \right)$, which is together with the inequality $\frac{1}{n} \sum_{k=1}^{n} M \left( \frac{|x_k - y_k|}{\rho} \right) < \varepsilon$, so the proof has been completed, that is, $x \overset{WM}{\sim} y$.

Theorem 2.8 If $M_1$ and $M_2$ are two Orlicz functions satisfying $\Delta_2$-condition, then the following statements hold:

If \( x^{M_2} y \) then \( x^{M_1+M_2} y \),

(ii) If \( x^{M_1+M_2} y \) then \( x^{M_1+M_2} y \).

**Proof** (i) If \( x^{M_2} y \) then there exists \( \rho > 0 \) such that

\[
\lim_{k \to \infty} M_2 \left( \frac{|y_k - L|}{\rho} \right) = 0. \tag{2}
\]

Choose \( \varepsilon > 0 \) and \( 0 < \delta < 1 \) such that \( M_1(u) < \varepsilon \) for every \( u \) with \( 0 \leq u \leq \delta \). Let us consider \( \lim_{k \to \infty} M_1(A_k) = \lim_{k,A_k \leq \delta} M_1(A_k) = \lim_{k,A_k \geq \delta} M_1(A_k) \), where \( A_k = M_2 \left( \frac{|y_k - L|}{\rho} \right) \). So we have

\[
\lim_{k,A_k \leq \delta} M_1 \left( M_2 \left( \frac{|y_k - L|}{\rho} \right) \right) \leq M_1(2) \lim_{k,A_k \geq \delta} M_2 \left( \frac{|y_k - L|}{\rho} \right). \tag{3}
\]

For \( A_k \geq \delta \), we have \( A_k < \frac{\Delta k}{\delta} < 1 + \frac{\Delta k}{\delta} \). Since \( M \) is non-decreasing and convex, it follows that,

\[
M_1(A_k) < M_1(1 + \frac{\Delta k}{\delta}) \leq M_1(2) + \frac{1}{2} M_1(\frac{\Delta k}{\delta}) \cdot \delta. \tag{4}
\]

and

\[
\lim_{k,A_k \geq \delta} M_1 \left( M_2 \left( \frac{|y_k - L|}{\rho} \right) \right) \leq \max \left( 1, K \delta^{-1} M_1(2) \right) \lim_{k,A_k \geq \delta} M_2 \left( \frac{|y_k - L|}{\rho} \right). \tag{5}
\]

From (2), (3), (4) and (5), we have \( \lim_{k \to \infty} M_1 \left( M_2 \left( \frac{|y_k - L|}{\rho} \right) \right) = 0 \). Hence \( x^{M_1+M_2} y \).

(ii) Suppose that \( x^{M_1+M_2} y \). Therefore,

\[
\lim_{k \to \infty} M_1 \left( \frac{y_k - L}{\rho} \right) = 0
\]

and

\[
\lim_{k \to \infty} M_2 \left( \frac{y_k - L}{\rho} \right) = 0.
\]

So the proof can be completed, since the equality

\[
(M_1 + M_2) \left( \frac{y_k - L}{\rho} \right) = M_1 \left( \frac{y_k - L}{\rho} \right) + M_2 \left( \frac{y_k - L}{\rho} \right)
\]

holds.

**Theorem 2.9** If \( M \) is an Orlicz function and \( \theta \) is a lacunary sequence, then the following statements hold:

(i) If \( \lim \sup_{r \to \infty} q_r < \infty \) then \( x^{N^M} y \) implies \( x^{W^M} y \)

(ii) If \( \lim \inf_{r \to \infty} q_r > 1 \) then \( x^{W^M} y \) implies \( x^{N^M} y \)

(iii) If \( 1 < \lim \inf_{r \to \infty} q_r \leq \lim \sup_{r \to \infty} q_r < \infty \), then \( x^{W^M} y \Leftrightarrow x^{N^M} y \).

**Proof**

(i) If \( \lim \sup_{r \to \infty} q_r < \infty \) then there exists \( K > 0 \) such that \( q_r < K \) for every \( r \). Now suppose that \( x^{N^M} y \) and \( \varepsilon > 0 \). There exists \( m_0 \) such that for every \( m \geq m_0 \),

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We can also find $R > 0$ such that $H_m \leq R$ for all $m$. Let $n$ be any integer with $k_r \geq n > k_{r-1}$. Now from the inequalities

$$\frac{1}{n} \sum_{k=1}^{n} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right) \leq \frac{1}{k_{r-1}} \sum_{k=1}^{k_r} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right)$$

$$= \frac{1}{k_{r-1}} \sum_{k=1}^{m_0} \sum_{k \in I_m} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right) +$$

$$+ \frac{1}{k_{r-1}} \sum_{k=m_0}^{k_r} \sum_{k \in I_m} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right)$$

$$\leq \frac{1}{k_{r-1}} \sum_{k=1}^{m_0} \sum_{k \in I_m} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right) + \frac{\varepsilon}{k_{r-1}} (k_r - k_{m_0})$$

$$\leq \frac{1}{k_{r-1}} \sup_{1 \leq k \leq m_0} H_k k_{m_0} + \varepsilon K$$

$$\leq \frac{R}{k_{r-1}} k_{m_0} + \varepsilon K$$

we complete the proof, that is, $x^{\omega M} y$.

(ii) Suppose that $x^{\omega M} y$. If $\liminf q_r > 1$ then there exists a $\delta > 0$ such that $q_r = \left( \frac{k_r}{k_{r-1}} \right) \geq 1 + \delta$ for sufficiently large $r$ which implies $\left( \frac{h_r}{q_r} \right) \geq \frac{\delta}{\delta + 1}$.

Hence we get

$$\frac{1}{k_r} \sum_{k=1}^{k_r} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right) \geq \frac{1}{k_r} \sum_{k \in I_r} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right)$$

$$= \frac{h_r}{k_r h_r} \sum_{k \in I_r} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right)$$

$$\geq \frac{\delta}{\delta + 1} \frac{1}{k_r} \sum_{k \in I_r} M \left( \left| \frac{x_k - y_k}{\rho} \right| \right)$$

which proves the claim, that is, $x^{\omega M} y$.

(iii) This immediately follows from (i) and (ii).

**Theorem 2.10** If $M$ is an Orlicz function and $\theta$ is a lacunary sequence, then

(i) If $x^{\omega M} y$ then $x^{\omega \theta} y$.

(ii) If $x^{\omega \theta} y$ and $x, y$ are bounded sequences, then $x^{\omega M} y$.
(iii) If \( x \) and \( y \) are bounded sequences then \( x \overset{N}{\preceq} y \Longleftrightarrow x \overset{M}{\prec} y \).

**Proof**

(i) If \( x \overset{M}{\prec} y \), then there exists \( \rho > 0 \) such that

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{R}_r} M\left(\frac{|x_k - L|}{\rho}\right) = 0.
\]

For \( \varepsilon > 0 \) with the equality

\[
\frac{1}{h_r} \sum_{k \in \mathbb{R}_r} M\left(\frac{|x_k - L|}{\rho}\right) = \frac{1}{h_r} \sum_{1} M\left(\frac{|x_k - L|}{\rho}\right) + \frac{1}{h_r} \sum_{2} M\left(\frac{|x_k - L|}{\rho}\right),
\]

the first summation is over \( \frac{|x_k - L|}{y_k} \geq \varepsilon \) and the second one over \( \frac{|x_k - L|}{y_k} < \varepsilon \). Therefore,

\[
\frac{1}{h_r} \sum_{k \in \mathbb{R}_r} M\left(\frac{|x_k - L|}{\rho}\right) \geq \frac{1}{h_r} \sum_{1} M\left(\frac{|x_k - L|}{\rho}\right) \geq M\left(\frac{\varepsilon}{\rho} \right)\left|\{ k \in I_r: \frac{|x_k - L|}{y_k} \geq \varepsilon \} \right|.
\]

This completes the proof.

(ii) Suppose that \( x \) and \( y \) are two bounded sequences and \( x \overset{N}{\preceq} y \). For some \( \rho > 0 \) there exists \( D > 0 \) such that \( \frac{|x_k - L|}{y_k} \leq D \) for all \( k \). For \( \varepsilon > 0 \) we have the equality

\[
\frac{1}{h_r} \sum_{k \in \mathbb{R}_r} M\left(\frac{|x_k - L|}{\rho}\right) = \frac{1}{h_r} \sum_{1} M\left(\frac{|x_k - L|}{\rho}\right) + \frac{1}{h_r} \sum_{2} M\left(\frac{|x_k - L|}{\rho}\right).
\]

Here, the first summation is over \( \frac{|x_k - L|}{y_k} \geq \varepsilon \) and the second one over \( \frac{|x_k - L|}{y_k} < \varepsilon \). Therefore,

\[
\frac{1}{h_r} \sum_{k \in \mathbb{R}_r} M\left(\frac{|x_k - L|}{\rho}\right) \leq M(D) \frac{1}{h_r} \left|\{ k \in I_r: \frac{|x_k - L|}{y_k} \geq \varepsilon \} \right| + M\left(\frac{\varepsilon}{\rho}\right).
\]

This completes the proof.

(iii) Follows from (i) and (ii).

**Theorem 2.11** If \( M \) is an Orlicz function and \( \theta = (k_r) \) is a lacunary sequence, then the following statements hold:

(i) If \( \limsup_r q_r < \infty \) then \( x \overset{M}{\preceq} y \) implies \( x \overset{M}{\prec} y \).

(ii) If \( \liminf_r q_r > 1 \) then \( x \overset{M}{\prec} y \) implies \( x \overset{M}{\preceq} y \).

(iii) If \( 1 < \liminf_r q_r \leq \limsup_r q_r < \infty \), then \( x \overset{M}{\preceq} y \Longleftrightarrow x \overset{M}{\prec} y \)

**Proof**
(i) If \( \limsup_{r} q_{r} < \infty \) then there exists \( K > 0 \) such that \( q_{r} < K \) for every \( r \).

Now suppose that \( x \overset{M}{\lesssim}_{y} y \) and take \( A_{r} = \left\{ r \in I_{r}; M \left( \frac{|x_{k} - L|}{y_{k}} \right) \geq \varepsilon \right\} \). There is an \( r_{0} \) such that \( \frac{h_{r}}{k_{r}} < \varepsilon \) for all \( r > r_{0}, \varepsilon > 0 \).

Now let \( B = \max\{A_{r}; 1 < r < r_{0}\} \) and let \( n \) be any integer satisfying \( k_{r} \geq n > k_{r-1} \), then we have the inequalities

\[
\frac{1}{n} \left\{ k \leq n; M \left( \frac{|x_{k} - L|}{y_{k}} \right) \geq \varepsilon \right\} \leq \frac{1}{k_{r-1}} \left\{ k \leq k_{r}; M \left( \frac{|x_{k} - L|}{y_{k}} \right) \geq \varepsilon \right\} \\
= \frac{1}{k_{r-1}} \{ A_{1} + A_{2} + \ldots + A_{n} + \ldots A_{r} \} \leq \frac{B}{k_{r-1}} k_{r-1} r_{0} + \frac{1}{k_{r-1}} \sum_{m=r_{0}+1}^{k_{r}} h_{m} A_{r} \leq \frac{B}{k_{r-1}} r_{0} + \frac{1}{k_{r-1}} \sup_{m} h_{m} \sum_{r_{0}+1}^{k_{r}} h_{m} k_{r-1} r_{0} + q_{r} \varepsilon \leq B \frac{k_{r} r_{0} - k_{r} r_{0}}{k_{r-1}} \leq B \frac{k_{r} r_{0}}{k_{r-1}} \varepsilon .
\]

which completes the proof, that is, \( x \overset{M}{\lesssim} y \).

(ii) Let \( \liminf_{r} q_{r} > 1 \). There exists \( \delta > 0 \) such that \( q_{r} = \left( \frac{k_{r}}{k_{r-1}} \right) \geq 1 + \delta \) for sufficiently large \( r \) which implies that

\[
\frac{h_{r}}{k_{r}} \geq \frac{\delta}{\delta + 1} .
\]

If \( x \overset{M}{\lesssim} y \) then for every \( \varepsilon > 0 \) and for sufficiently large, \( r \), we have

\[
\frac{1}{k_{r}} \left\{ k \leq k_{r}; M \left( \frac{|x_{k} - L|}{y_{k}} \right) \geq \varepsilon \right\} \geq \frac{1}{k_{r}} \left\{ k \in I_{r}; M \left( \frac{|x_{k} - L|}{y_{k}} \right) \geq \varepsilon \right\} \geq \frac{\delta}{\delta + 1} \frac{1}{h_{r}} \left\{ k \in I_{r}; M \left( \frac{|x_{k} - L|}{y_{k}} \right) \geq \varepsilon \right\} .
\]

which completes the proof.

(iii) This immediately follows from (i) and (ii).

If \( 0 < p \leq t \), then we have the following theorem.

**Theorem 2.12** If \( M \) is an Orlicz function and \( \theta = (k_{r}) \) is a lacunary sequence, then \( x^{N_{\theta}^{M_{t}}} \) \( y \) implies \( x^{N_{\theta}^{M_{p}}} \) \( y \).

**Proof** Suppose that \( x^{N_{\theta}^{M_{t}}} \) \( y \). It follows from Holder’s inequality,

\[
\frac{1}{h_{t}} \sum_{k \in I_{r}} \left[ M \left( \frac{|x_{k} - L|}{y_{k}} \right) \right]^{p} \leq \left[ \frac{1}{h_{t}} \sum_{k \in I_{r}} \left( \frac{|x_{k} - L|}{y_{k}} \right) \right]^{t} \left[ \frac{1}{h_{t}} \sum_{k \in I_{r}} \left( \frac{|x_{k} - L|}{y_{k}} \right) \right]^{p-t} .
\]

Thus we have \( x^{N_{\theta}^{M_{p}}} \) \( y \).
We now consider the sequences \( p = (p_k) \) and \( t = (t_k) \) which are not constants.

**Theorem 2.13** If \( M \) is an Orlicz function, \( \theta = (k_r) \) is a lacunary sequence, \( 0 < p_k \leq t_k \) for all \( k \) and \( \left( \frac{1}{p_k} \right) \) is the bounded, then \( x^{\frac{M(t)}{\frac{1}{p_k}}}_\theta \) \( y \) implies \( x^{\frac{M(p)}{\frac{1}{t_k}}}_\theta \) \( y \).

**Proof** Suppose that \( x^{\frac{M(t)}{\frac{1}{p_k}}}_\theta \) \( y \), \( z_k = \left[ M\left( \frac{1}{p_k} \right) \right]^{\frac{1}{t_k}} \lambda_{k-}(p_k/t_k) \). So we have \( 0 < \lambda \leq \lambda_k \leq 1 \). We define the sequences \((u_k)\) and \((v_k)\) as \( u_k = z_k \) if \( z_k \geq 1 \) and \( v_k = 0 \) if \( z_k < 1 \). If \( v_k = z_k \) and \( u_k = 0 \), then we have \( z_k = u_k + v_k \); \( z_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \); \( u_k^{\lambda_k} \leq u_k \leq z_k \) and \( v_k^{\lambda_k} \leq v_k \). Therefore,

\[
\frac{1}{h_r} \sum_{k \in r} z_k^{\lambda_k} = \frac{1}{h_r} \sum_{k \in r} (u_k^{\lambda_k} + v_k^{\lambda_k}) \\
\leq \frac{1}{h_r} \sum_{k \in r} z_k + \frac{1}{h_r} \sum_{k \in r} v_k^{\lambda_k}.
\]

Now for each \( r \), the inequalities

\[
\frac{1}{h_r} \sum_{k \in r} v_k^{\lambda_k} = \sum_{k \in r} \left( \frac{1}{h_r} v_k \right)^\lambda \left( \frac{1}{h_r} \right)^{1-\lambda} \\
\leq \left( \sum_{k \in r} \left( \frac{1}{h_r} v_k \right)^\lambda \right)^\lambda \left( \sum_{k \in r} \left( \frac{1}{h_r} \right)^{1-\lambda} \right)^{1-\lambda} \\
< \left( \frac{1}{h_r} \sum_{k \in r} v_k \right)^\lambda
\]

hold. Therefore, we have the inequalities

\[
\frac{1}{h_r} \sum_{k \in r} \left[ M\left( \frac{1}{p_k} \right) \right]^{p_k} z_k^{\lambda_k} = \frac{1}{h_r} \sum_{k \in r} z_k^{\lambda_k} \\
\leq \frac{1}{h_r} \sum_{k \in r} z_k + \frac{1}{h_r} \sum_{k \in r} v_k^{\lambda_k} \\
= \left\{ \begin{array}{ll}
\frac{1}{h_r} \sum_{k \in r} z_k + \left( \frac{1}{h_r} \sum_{k \in r} z_k \right)^\lambda, & z_k \geq 1 \\
\frac{1}{h_r} \sum_{k \in r} z_k + \left( \frac{1}{h_r} \sum_{k \in r} z_k \right)^\lambda, & z_k < 1 
\end{array} \right\} \\
\leq \left\{ \begin{array}{ll}
\frac{1}{h_r} \sum_{k \in r} z_k, & z_k \geq 1 \\
2 \left( \frac{1}{h_r} \sum_{k \in r} z_k \right)^\lambda, & z_k < 1 
\end{array} \right\}
\]

which completes the proof, that is, \( x^{\frac{M(p)}{\frac{1}{t_k}}}_\theta \) \( y \).
References


