Further Generalisation of Density

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Abstract:

The purpose of our present note is to further generalise the definition of lower asymptotic density given by Das, Mishra and Ray [3] and introduced by Freedman and Sember [4], using the class of conservative matrices \( \mathcal{A} = (a_{nk}(i)) \) such that \( \lim_n a_{nk}(i) = 0 \) uniformly in \( i \) which can be applicable to characterise the conservative matrices.

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1. Definitions and Notations:

Let \( \mathcal{A} = (A^i) \) be the sequence of matrices \( A^i = (a_{nk}(i)) \) of real entries and for a real number sequence \( x = (x_k) \), we write

\[
A^i_n(x) = \sum_k a_{nk}(i)x_k
\]

(1.1)

if it exists for each \( n \) and \( i \geq 0 \) (summation without limit is from 0 to \( \infty \)).

The sequence \( x \) is said to be summable to the value \( s \) by the method \( \mathcal{A} \) if

\[
\lim_{n \to \infty} A^i_n(x) = s \quad \text{uniformly in } i
\]

The method \( \mathcal{A} \) is conservative [5] if and only if the following conditions hold:

(i) \( \| \mathcal{A} \| = \sup_{n,i \geq 0} \sum_k |a_{nk}(i)| < \infty \)

(ii) \( \exists \ a_k \in \mathbb{C} : \lim_n a_{nk}(i) = a_k \) for fixed \( k \) uniformly in \( i \)

(iii) \( \exists a \in \mathbb{C} : \lim_n \sum_k a_{nk}(i) = a \) uniformly in \( i \)

The method \( \mathcal{A} \) is regular if further \( a_k = 0, \ a = 1 \). We write

\[
\alpha(\mathcal{A}) = a - \sum_k a_k
\]

(1.2)

The method \( \mathcal{A} \) is called conull if \( \alpha(\mathcal{A}) = 0 \), otherwise coregular.

We write,

\[
a_{nk}^+(i) = \max(a_{nk}(i), 0), \ a_{nk}^-(i) = \max(-a_{nk}(i), 0)
\]

so that

\[
\sum_k |a_{nk}(i)| \leq \sum_k a_{nk}^+(i) + \sum k a_{nk}^-(i)
\]

(1.3)

and

\[
\sum_k a_{nk}(i) = \sum_k a_{nk}^+(i) - \sum_k a_{nk}^-(i)
\]

(1.4)

The method \( \mathcal{A} \) is called almost positive if
2. Introduction:

In this paper, we have generalised the definition of lower asymptotic density given in [3] and then we have ensured this type of densities in the main results.

Here we have a new definition for a generalisation of density.

Definition 2.1: Let $0 < \Omega < \infty$

A function $\delta: P(N) \rightarrow [0, \Omega]$ is called a lower $\Omega$-asymptotic density if the following axioms hold for all $E, F \subseteq P(N)$.

\begin{align*}
D_1: & \text{ } E \triangle F \text{ is finite } \Rightarrow \delta(E) = \delta(F) \\
D_2: & \text{ } E \cap F = \emptyset \Rightarrow \delta(E) + \delta(F) \leq \delta(E \cup F) \\
D_3: & \text{ } \delta(E) + \delta(F) \leq \Omega + \delta(E \cap F) \\
D_4: & \text{ } \delta(N) = \Omega
\end{align*}

The upper $\Omega$-asymptotic-density $\overline{\delta}$ associated with $\delta$ is defined as

$$\overline{\delta}(E) = \Omega - \delta(E')$$

where $E'$ is the complement of $E$.

In case $\delta(E) = \overline{\delta}(E)$, then $\delta$ is called $\Omega$-asymptotic density or simply $\Omega$-density. In the case $\Omega = 1$, $\Omega$-density reduces to usual asymptotic density or just density which is due to Freedman and Sember [4].

Let $\sigma$ be a one-to-one linear mapping of the set $N$ of natural numbers into itself. Let $X_E$ be the characteristic function of the set $E$. Then for $E \subseteq \sigma(P(N))$

$$\delta_*(E) = \liminf_n \frac{1}{n} \sum_{k=1}^{\sigma(n)} X_{E'}(k)$$

provides an example of lower $\Omega$-asymptotic density as it can be easily verified that conditions $D_1$-$D_4$ stated above are fulfilled by $\delta_*(E)$.

The upper $\Omega$-asymptotic density is given by

$$\overline{\delta}_*(E) = \Omega - \delta_*(E')$$

$$= \Omega - \liminf_n \frac{1}{n} \sum_{k=1}^{\sigma(n)} X_{E'}(k)$$

$$= \limsup_n \frac{1}{n} \sum_{k=1}^{\sigma(n)} X_{E}(k)$$

$$= \delta^*(E) \text{ (say)}$$

We have the $\Omega$-density if $\delta_*(E) = \delta^*(E)$, and in this case, the $\Omega$-density is denoted by $\delta(E)$ and is given by

$$\delta(E) = \lim_n \frac{1}{n} \sum_{k=1}^{\sigma(n)} X_{E}(k)$$

If $\sigma = 2n + 1$, then $\delta(E)$ in (2.4) is a 2-density.

3. Lemmas:

Before we proceed to prove some theorems, we require the following lemmas.

Lemma 3.1: ([1], Theorem 2(a))

Let $\mathcal{A} = \{a_{nk}(i)\}$ be conservative. Then

$$\limsup_n \frac{1}{n} \sum_k (a_{nk}(i) - a_k) x_k \leq \frac{[\alpha(\mathcal{A}) + \alpha(\mathcal{A})]}{2} \limsup_n x - \frac{[\alpha(\mathcal{A}) - \alpha(\mathcal{A})]}{2} \liminf_n x$$

if and only if

$$\limsup_n \frac{1}{n} \sum_k |a_{nk}(i) - a_k| = |\alpha(\mathcal{A})|$$
Lemma 3.2: 
Let \( \mathcal{A} = (a_{nk}(i)) \) be conservative. Then 
\[
\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leq \alpha(\mathcal{A}) \limsup x
\] 
(3.3)
if and only if 
\[
\limsup_n \sup_i \sum_k |(a_{nk}(i) - a_k)| = \alpha(\mathcal{A})
\] 
(3.4)

Proof: 
By definition 
\[
\sum_k |a_{nk}(i) - a_k| = \sum_k (a_{nk}(i) - a_k)^+ + \sum_k (a_{nk}(i) - a_k)^-
\] 
(3.5)
\[
\sum_k (a_{nk}(i) - a_k) = \sum_k (a_{nk}(i) - a_k)^+ - \sum_k (a_{nk}(i) - a_k)^-
\] 
(3.6)
By definition of almost positive method and identities (3.5) and (3.6) it follows that the above method \((a_{nk}(i) - a_k)\) is almost positive if and only if 
\[
\limsup_n \sup_i \sum_k |a_{nk}(i) - a_k| = \alpha(\mathcal{A})
\]

Now it follows from Lemma 3.1 (since \(\alpha(\mathcal{A}) > 0\)) that the inequality holds.

Lemma 3.3: 
Let \( \mathcal{A} \) be conservative. Then 
\[
\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leq \alpha(\mathcal{A})
\]
(3.7)
if and only if (3.4) holds.

Proof: 
This follows from Lemma 3.2 by taking \( x_n = x_k(n) \) and noting that \( \limsup_n x_k(n) = 1 \)

Lemma 3.4: 
Let \( \|\mathcal{A}\| < \infty \), then 
\[
\limsup_n \sup_i \sum_k a_{nk}(i)x_k \leq \alpha(\mathcal{A}) \limsup x
\]
(3.8)
if and only if \( \alpha(\mathcal{A}) = a \) and 
\[
\limsup_n \sup_i \sum_k |a_{nk}(i)| = a
\]
(3.9)

Proof: 
By definition of almost positive method and identities (1.3) and (1.4) it follows that the above method \( \mathcal{A} = (a_{nk}(i)) \) is almost positive if and only if 
\[
\limsup_n \sup_i \sum_k |a_{nk}(i)| = a = \alpha(\mathcal{A})
\]

Now it follows from Lemma 3.1 (since \( \alpha(\mathcal{A}) > 0 \)) that the inequality holds.

Lemma 3.5: 
Let \( \|\mathcal{A}\| < \infty \), then 
\[
\limsup_n \sum_i \sum_k a_{nk}(i)x_k \leq a
\]
(3.10)
if and only if \( \alpha(\mathcal{A}) = a \) and \( \mathcal{A} \) is almost positive i.e., (3.9) holds.

Proof: 
This follows from Lemma 3.4 by taking \( x_n = x_k(n) \) and noting that \( \limsup_n x_k(n) = 1 \)

4. Main Results: 
Now we obtain some \( \Omega \)-generalised density associated with a sequence of matrices in the following theorems for \( \Omega = a \) or \( \Omega = \alpha(A) \) which are noted earlier. For this we again generalise the density given in [2] and [3].
Theorem 4.1: 
Let $\mathcal{A} = (a_{nk}(i))$ be conservative such that $\lim_n a_{nk}(i) = 0$ uniformly in $i$. Let $\mathcal{A}$ be almost positive. Then for $E \subseteq N$

(a) 

$$d_{\mathcal{A}}(E) = \liminf_n \inf_1 \sum_k a_{nk}(i) \chi_E(k)$$

(4.1)

is a lower $\alpha$-asymptotic density 

(b) The upper $\alpha$-asymptotic density is given by 

$$\tilde{d}_{\mathcal{A}}(E) = \limsup_n \sup_1 \sum_k a_{nk}(i) \chi_E(k)$$

(4.2)

(c) The density $d_{\mathcal{A}}(E)$ is ensured when $d_{\mathcal{A}}(E) = d_{\mathcal{A}}(E) = \tilde{d}_{\mathcal{A}}(E)$, that is 

$$d_{\mathcal{A}}(E) = \lim_n \sum_k a_{nk}(i) \chi_E(k)$$

uniformly in $i$

(4.3)

is a $\alpha -$asymptotic density.

Proof: Since $\mathcal{A} = (a_{nk}(i))$ be conservative, almost positive and $\lim_n a_{nk}(i) = 0$ for each $k$, so

$$\lim_n \sum_k |a_{nk}(i)| = \lim_n \sum_k a_{nk}(i) = a$$ 

uniformly in $i$ and 

$$\alpha(\mathcal{A}) = a > 0$$ 

(4.4)

To prove that $d_{\mathcal{A}}(E)$ is a lower $\alpha$-density we have to prove that $d_{\mathcal{A}}(E)$ satisfies the following conditions (i) –(iv) by taking $\Omega = a = \alpha(\mathcal{A})$ in Definition 2.1 i.e.,

(i): $E \Delta F$ is finite $\Rightarrow d_{\mathcal{A}}(E) = d_{\mathcal{A}}(F)$

(ii): $E \cap F = \emptyset \Rightarrow d_{\mathcal{A}}(E) + d_{\mathcal{A}}(F) \leq (d_{\mathcal{A}}E \cup F)$

(iii) For all $E \& F$, $d_{\mathcal{A}}(E) + d_{\mathcal{A}}(F) \leq a + d_{\mathcal{A}}(E \cap F)$

(iv) $d_{\mathcal{A}}(N) = a$

Here if 

$$E \Delta F \text{ finite } \Rightarrow \chi_E(k) = \chi_F(k)$$

$k > K$ for some integer $K$

$$\Rightarrow \sum_{k=1}^K |a_{nk}(i)| (\chi_E(k) - \chi_F(k)) = \sum_{k=1}^K a_{nk}(i) (\chi_E(k) - \chi_F(k))$$

(4.5)

The expression (4.5) is bounded by $\sum_{k=1}^K |a_{nk}(i)|$ which converges to $0$ as $n \to \infty$ uniformly in $i$.

Hence it follows that 

$$d_{\mathcal{A}}(E) = d_{\mathcal{A}}(F)$$

This proves the requirement of (i). Next

$$E \cap F = \emptyset \Rightarrow \chi_{E \cap F} = \chi_E + \chi_F$$

Since the functional $d_{\mathcal{A}}(E)$ is super additive, we have 

$$d_{\mathcal{A}}(E \cup F) \geq d_{\mathcal{A}}(E) + d_{\mathcal{A}}(F)$$

(4.6)

This proves (ii).

Since, $\chi_{E \cup F} = \chi_E + \chi_F - \chi_{E \cap F}$; we have

$$a + d_{\mathcal{A}}(E \cap F) \geq a + \liminf_n \inf_1 \sum_k a_{nk}(i) \chi_E(k)$$

$$+ \liminf_n \inf_1 \sum_k a_{nk}(i) \chi_F(k)$$

$$+ \liminf_n \inf_1 (-\sum_k a_{nk}(i) \chi_{E \cup F}(k))$$

$$= a + d_{\mathcal{A}}(E) + d_{\mathcal{A}}(F) - \tilde{d}_{\mathcal{A}}(E \cup F)$$

$$\geq (d_{\mathcal{A}}E) + d_{\mathcal{A}}(F)$$

(4.7)

$$(as \ a - \tilde{d}_{\mathcal{A}}(E \cup F) \geq 0 \ by \ Lemma \ 3.5)$$

$$\Rightarrow (d_{\mathcal{A}}E) + d_{\mathcal{A}}(F) \leq a + d_{\mathcal{A}}(E \cap F)$$


This proves (iii).

Lastly,
\[
d_{\mathcal{A}}(N) = \liminf_n \inf_i \sum_k a_{nk}(i) \chi_k(k)
\]
\[
= \liminf_n \inf_i \sum_k a_{nk}(i)
\]
\[
= a
\]

So (iv) holds

This proves Theorem 1(a).

(b) Now since \( \chi_{E^c} = 1 - \chi_E \), by Theorem 1(a) and (2.1) it follows that
\[
\tilde{d}_{\mathcal{A}}(E) = a - d_{\mathcal{A}}(E^c)
\]
\[
= a - \liminf_n \inf_i \sum_k a_{nk}(k)(1 - \chi_E(k))
\]
\[
= \limsup_n \sup_i \sum_k a_{nk}(i) \chi_E(k)
\]

which is the corresponding upper \( a \)- asymptotic density

This proves Theorem 1(b).

In the case, \( d_{\mathcal{A}}(E) = \tilde{d}_{\mathcal{A}}(E) \), then it can be easily shown that, Theorem 1(c) holds.

**Theorem 4.2:**

Let \( \mathcal{A} = (a_{nk}(i)) \) be coregular and almost positive method with \( a(\mathcal{A}) = a - \sum_k a_k \).

Let \( \mathcal{B} = (b_{nk}(i)) = (a_{nk}(i) - a_k) \) be almost positive.

Then for \( E \subseteq \mathbb{N} \).

(a) \( \tilde{d}_{\mathcal{B}}(E) = \liminf_n \inf_i \sum_k b_{nk}(i) \chi_E(k) \) is a lower \( \alpha(\mathcal{A}) \)-asymptotic density.

(b) The upper \( \alpha(\mathcal{A}) \)-asymptotic density is
\[
\tilde{d}_{\mathcal{B}}(E) = \limsup_n \sup_i \sum_k b_{nk}(i) \chi_E(k)
\]

(c) The density \( d_{\mathcal{B}}(E) \) is ensured when \( d_{\mathcal{B}}(E) = \tilde{d}_{\mathcal{B}}(E) = \tilde{d}_{\mathcal{A}}(E) \), that is
\[
d_{\mathcal{B}}(E) = \lim_n \sum_k b_{nk}(i) \chi_E(k)
\]

is a \( \alpha(\mathcal{A}) \)-asymptotic density.

**Proof:**

By definition,
\[
\lim_n \sum_k b_{nk}(i) = \lim_n \sum_k (a_{nk}(i) - a_k) = \alpha(\mathcal{A})
\]
uniformly in \( i \) and
\[
\lim_n b_{nk}(i) = \lim_n (a_{nk}(i) - a_k) = 0
\]
uniformly in \( i \), since \( \alpha(\mathcal{A}) \neq 0 \). Lastly
\[
\sup_{n,i} \sum_k |b_{nk}(i)| \leq \sum_k |a_{nk}(i)| + \sum_k |a_k| < \infty
\]

So we can conclude that \( \mathcal{B} = (b_{nk}(i)) = (a_{nk}(i) - a_k) \) is a conservative method with \( \lim_n b_{nk}(i) = 0 \) uniformly in \( i \).

Also it is given that \( \mathcal{B} \) is almost positive.

Since \( \mathcal{A} \) is co-regular and almost positive that implies
\[
\alpha(\mathcal{B}) = \lim_n \sum_k b_{nk}(i) - \sum_k \lim b_{nk}(i)
\]
\[
= \lim_n (a_{nk}(i) - a_k)
\]
(by (4.10))
\[
= \alpha(\mathcal{A})
\]

So \( \alpha(\mathcal{B}) = \alpha(\mathcal{A}) > 0 \) \((\because \mathcal{B} \text{ is almost positive})\)
To prove that $d_B(E)$ is a lower $\alpha(\mathcal{A})$-density we have to prove that $d_B(E)$ satisfies the following conditions (i) – (iv) by taking $\Omega = \alpha(\mathcal{A})$ in definition 2.1 i.e.,

(i) $E \Delta F$ is finite $\Rightarrow d_B(E) = d_B(F)$

(ii) $E \cap F = \emptyset \Rightarrow d_B(E) + d_B(F) \leq d_B(E \cup F)$

(iii) For all $E \& F$, $d_B(E) + d_B(F) \leq \alpha(\mathcal{A}) + d_B(E \cap F)$

(iv) $d_B(N) = \alpha(\mathcal{A})$

By applying the technique of Theorem 4.1 of the present work to $\mathcal{B}$ and $d_B(E)$ we can establish Theorem 4.2 as $\mathcal{B} = (b_{nk}(i)) = (a_{nk}(i) - a_k)$ be almost positive with $\lim_{n} b_{nk}(i) = 0$ uniformly in $i$.

**Corollary 4.2.1:**

Let $\mathcal{A} = (a_{nk}(i))$ is regular and almost positive method. Then for $E \subseteq N$

$$\delta_{\mathcal{A}}(E) = \liminf_n \inf_i \sum_k a_{nk}(i) \chi_E(k)$$

is a lower asymptotic density

**Proof:** In a regular and almost positive method , $\alpha(\mathcal{A}) = 1$

So by Theorem 1 we get $\delta_{\mathcal{A}}(E)$ satisfies all the conditions $D_1$-$D_4$ to be a lower $\Omega$-density for $\Omega = 1$

Hence $\delta_{\mathcal{A}}(E)$ is a lower 1-density or just a lower density. (Also see[2],[3])

**Example 4.1:** In the case

$$a_{nk}(i) = \begin{cases} \frac{c}{n+1} & i \leq k \leq i + n \\ 0 & \text{otherwise} \end{cases}$$

we get c-density where c is any fixed constant.

**References:**


