Abstract

This paper proposes the least-squares (LS) finite impulse response (FIR) smoother estimating the signal at the start time of the fixed interval in the FIR smoother and filter in linear discrete-time wide-sense stationary stochastic systems. It is assumed that the signal is observed with additional white noise and is uncorrelated with the observation noise. The LS FIR smoothing estimate is given as a linear convolution of the impulse response function data and the observed values. By solving the simultaneous linear equations transformed from the Wiener-Hopf equation, we can calculate the finite number of impulse response function data. The necessary information in the LS FIR smoothing algorithm is the auto-covariance function data of the signal process, $K(i), 1 \leq i \leq L$, for the finite interval $L$, the variance of the signal process and the variance of the observation noise process. The auto-covariance function data of the signal process, $K(i), 1 \leq i \leq L$, are equal to the auto-covariance function data of the observation process, $K_y(i), 1 \leq i \leq L$. This paper also proposes the Levinson-Durbin algorithm, which needs less arithmetic operations than the Gauss-Jordan elimination method used in the matrix inversion, for the impulse response function data in the LS FIR smoothing problem. The proposed LS FIR smoother estimating the signal at the start time of the fixed interval and filter are numerically compared.

Keywords: LS FIR smoother; Covariance information; Wiener-Hopf equation; Levinson-Durbin algorithm; Discrete-time stochastic systems.

1. Introduction

Concerning the finite impulse response (FIR) estimation problems, there are studies on the FIR filter [1], [2], the receding horizon Kalman FIR filter [3]-[5], the FIR smoother [6] and the $H_\infty$ FIR smoother [7]. In linear discrete-time stochastic systems, the recursive least-squares (RLS) Wiener FIR filter [8] calculates the filtering estimate in terms of the finite number of observed values. In [9], the RLS Wiener FIR fixed-lag smoother and filter are proposed.

In the approach along with the Kalman filter with the state-space models, the RLS fixed-lag smoothing algorithm is devised by introducing the augmentation in the state equation [10]. Alternatively, the RLS Wiener fixed-lag smoother is proposed [11], [12] in linear discrete-time stochastic systems. The RLS Wiener fixed-lag smoothing algorithm is derived by using the invariant imbedding method from the Wiener-Hopf equation. In [13], the LS FIR smoother estimating the signal at the start time of the fixed interval is proposed in linear continuous-time stochastic systems. However, the LS FIR smoother estimating the signal at the start time of the fixed interval is not developed in linear discrete-time stochastic systems.
In [9], the RLS FIR Wiener fixed-lag smoothing and filtering algorithms are presented in linear discrete-time stochastic systems. In the equation transformed from the Wiener-Hopf equation in the RLS FIR fixed-lag smoothing problem, due to the auto-covariance function $K(k-\text{lag},s)$ of the signal process for $1 \leq s \leq k$, the derivation of the estimation algorithms, based on the invariant embedding method, is not easy to be derived. Here, lag represents the fixed lag. To avoid this trouble, in [9], the auto-covariance function $K(k-\text{lag},s)$ is expressed in terms of $K(k,s), K(k+1,s), K(k+2,s), \ldots, K(k+\text{lag},s), 1 \leq s \leq k$. Through this step, the RLS Wiener FIR fixed-lag smoothing and filtering algorithms are derived. However, there remains a problem, as lag becomes large, the expression of $K(k-\text{lag},s)$ in terms of $K(k,s), K(k+1,s), K(k+2,s), \ldots, K(k+\text{lag},s), 1 \leq s \leq k$, becomes complicated. The RLS Wiener FIR fixed-lag smoothing and filtering algorithms [9] require the information of the observation vector $H$, the system matrix $\Phi$ for the state vector $x(k)$ and the variance $K_x(k,k)=K_x(0)$ of the state vector. The system matrix is obtained by using the auto-covariance data of the signal process in the relation with the signal autoregressive (AR) model of order $N$.

Without using the complicated expressions in the auto-covariance function $K(k-\text{lag},s)$ for the large value of lag in [9], this paper proposes the least-squares (LS) FIR smoother estimating the signal at the start time of the fixed interval $L$, in linear discrete-time wide-sense stationary stochastic systems. It is assumed that the signal is observed with the additional white noise process, and the signal process is uncorrelated with the observation noise process. This paper, in Theorem 1, based on the simultaneous linear equations transformed from the Wiener-Hopf equations in the LS FIR smoothing and filtering problems, calculates the finite impulse response function data. The FIR smoothing estimate is given as a linear convolution of the impulse response function data and the observed values. The necessary information in calculating the LS FIR smoothing estimate $\hat{z}(k-L,k)$ is the auto-covariance data of the signal process, $K(i), 1 \leq i \leq L$, the variance of the signal process, $K(0)$, and the variance of the observation noise process, $R$.

It is also shown that the finite impulse response function data for the filtering estimate are given as the reverse order data of those in the FIR smoother estimating the signal at the start time of the fixed interval. The impulse response function data for the smoothing estimate are calculated by applying the Gauss-Jordan elimination method to the simultaneous linear equations. For the $L+1$ number of simultaneous linear equations, the Gauss-Jordan elimination method needs the order of $O\left((L+1)^3\right)$ arithmetic operations. In the matrix and vector expression of the simultaneous linear equations, there appears the $(L+1)$ by $(L+1)$ square Toeplitz matrix. From the property of the Toeplitz matrix, for the calculation of the impulse response function data, the Levinson-Durbin algorithm is derived. The Levinson-Durbin algorithm requires the order of $O\left((L+1)^2\right)$ arithmetic operations [14] and is computationally faster than the Gauss-Jordan elimination method in the matrix inversion. Theorem 2 proposes the Levinson-Durbin algorithm for the optimal impulses response function data in the FIR smoothing problem estimating the signal at the start time of the fixed interval. Here, the Toeplitz matrix is expressed in terms of the auto-covariance function data of the observation process, $K_y(i), 1 \leq i \leq L$, and the variance of the observation process, $K_y(0)$, instead of the auto-covariance function data of the signal process, $K(i), 1 \leq i \leq L$, the variance of the signal process, $K(0)$, and the variance of the observation noise, $R$.

Section 4 shows the FIR smoothing error variance function to validate the stability of the proposed LS FIR smoothing algorithm. Section 5 demonstrates a numerical simulation example to show the estimation characteristics of the proposed LS FIR smoother and filter.

### 2. LS FIR Smoothing and Filtering Problem

Let the scalar observation equation and the state equation for the state vector $x(k)$ be expressed by

$$\begin{align*}
y(k) &= z(k) + \nu(k), \quad z(k) = Hx(k), \\
x(k+1) &= \Phi x(k) + w(k)
\end{align*} \tag{1}$$
in linear discrete-time wide-sense stationary stochastic systems. Here, \( z(k) \) represents the signal, \( v(k) \) the white observation noise, \( x(k) \) the \( n \)-dimensional state vector, \( H \) the \( 1 \times n \) observation matrix, \( \Phi \) the system matrix or the state-transition matrix and \( w(k) \) the white input noise. It is assumed that the observation noise process and the input noise process are mutually independent. Also, it is assumed that the signal process and the observation noise process are mutually independent and have zero means. Let the auto-covariance functions of \( v(k) \) and \( w(k) \) be given by

\[
E\left[ v(k)v^T(s) \right] = R\delta_k (k-s), \quad R > 0,
\]

\[
E\left[ w(k)w^T(j) \right] = Q\delta_k (k-j), \quad Q > 0.
\]

Here, \( \delta_k (k-j) \) denotes the Kronecker \( \delta \) function. Let \( K(k,s) = K(k-s) \) represent the auto-covariance function of the signal process in the wide-sense stationary stochastic systems [15], and let \( K(k,s) \) be expressed in the semi-degenerate kernel form of

\[
K(k,s) = \begin{cases} 
  A(k)B^T(s), & 0 \leq s \leq k, \\
  B(s)A^T(k), & 0 \leq k \leq s.
\end{cases}
\]

Here, along with the state equations in (1), \( A(k) \) and \( B^T(s) \) are expressed as \( A(k) = H\Phi^k \) and \( B^T(s) = \Phi^{-j}K_y(s,s)H^T \). \( K_y(s,s) \) represents the variance function of the state vector \( x(k) \) and is equal to \( K_y(0) \) in the wide-sense stationary stochastic systems. Let the FIR smoothing estimate \( \hat{z}(k-L,k) \) of \( z(k-L) \) be given by

\[
\hat{z}(k-L,k) = \sum_{i=k-L}^{k} h(k,i)y(i),
\]

as a linear convolution of the impulse response function \( h(k,i) \) and the observed values \( y(i), k-L \leq i \leq k \). Let the filtering estimate \( \hat{z}(k,k) \) of \( z(k) \) be given by

\[
\hat{z}(k,k) = \sum_{i=k-L}^{k} h_f(k,i)y(i).
\]

We consider the estimation problem, which minimizes the mean-square value (MSV)

\[
J = E[\| (z(k-L) - \hat{z}(k-L,k)) \|^2]
\]

of the FIR smoothing errors. From an orthogonal projection lemma [15],

\[
z(k-L) - \sum_{i=k-L}^{k} h(k,i)y(i) \perp y(s), \quad k-L \leq i \leq k,
\]

the Wiener-Hopf equation for the impulse response function is given by

\[
E\left[ z(k-L)y(s) \right] = \sum_{i=k-L}^{k} h(k,i)K_y(i,s).
\]

Here ‘ \( \perp \) ’ denotes the notation of the orthogonality and \( K_y(i,s) \) represents the auto-covariance function of the observation process. Substituting (1) and (2) into (8), we obtain

\[
h(k,s)R = K(k-L,s) - \sum_{i=k-L}^{k} h(k,i)K(i,s), \quad k-L \leq s \leq k.
\]
In the auto-covariance function $K(k-L, s)$ of the signal process, the variable $s$ takes the values of $k-L \leq s \leq k$.

Hence, in the case of large value of $L$, the derivation of the RLS Wiener FIR smoothing algorithm might be complicated. In section III, let us show on the calculation of the finite number of optimal impulse response function in (9). Then (4) calculates the LS FIR smoothing estimate as a linear convolution of the impulse response function data and the observed values.

Similarly, in the FIR filtering problem, the impulse response function $h_f(k, s)$ satisfies

$$h_f(k, s) = K(k, s) - \sum_{i=k-L}^{k} h_f(k, i) K(i, s), \quad k-L \leq s \leq k. \quad (10)$$

### 3. LS Impulse Response Function and FIR Smoothing Estimate

Now, let us obtain, from (8), the simultaneous linear equations, which the optimal impulses response function satisfies, in the fixed-lag smoothing problem. Substituting $s = k-L$ into (9), we have

$$h(k, k-L) R = K(k-L, k-L) - (h(k, k-L) K(k-L, k-L)$$
$$+ h(k, k-L+1) K(k-L+1, k-L) + h(k, k-L+2) K(k-L+2, k-L) + \cdots$$
$$+ h(k, k) K(k, k-L). \quad (11)$$

For $s = k-L+1$, (9) is written as follows.

$$h(k, k-L+1) R = K(k-L, k-L+1) - (h(k, k-L) K(k-L, k-L+1)$$
$$+ h(k, k-L+1) K(k-L+1, k-L+1) + h(k, k-L+2) K(k-L+2, k-L+1)$$
$$+ \cdots + h(k, k) K(k, k-L+1). \quad (12)$$

For $s = k-L+2$, (9) is written as follows.

$$h(k, k-L+2) R = K(k-L, k-L+2) - (h(k, k-L) K(k-L, k-L+2)$$
$$+ h(k, k-L+1) K(k-L+1, k-L+2) + h(k, k-L+2) K(k-L+2, k-L+2) + \cdots + h(k, k) K(k, k-L+2). \quad (13)$$

Similarly, the substitutions of the variable $s$ continues for $s = k-L+3, k-L+4, \cdots, k-1, k$. Substitution of $s = k$ into (9) yields

$$h(k, k) R = K(k-L, k) - (h(k, k-L) K(k-L, k)$$
$$+ h(k, k-L+1) K(k-L+1, k) + \cdots + h(k, k) K(k, k). \quad (14)$$

In the wide-sense stationary stochastic systems, the auto-covariance function of the signal process satisfies the relationship $K(k, s) = K(s, k) = K(k-s) = K(s-k)$. By applying this relationship to (11)-(14), we have the simultaneous linear equations in matrix form as follows.

$$
\begin{bmatrix}
K_y(0) & K_y(1) & \cdots & K_y(L-1) & K_y(L) \\
K_y(1) & K_y(0) & \cdots & K_y(L-2) & K_y(L-1) \\
K_y(2) & K_y(1) & \cdots & K_y(L-3) & K_y(L-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_y(L) & K_y(L-1) & \cdots & K_y(1) & K_y(0)
\end{bmatrix}
\begin{bmatrix}
h(k, k-L) \\
h(k, k-L+1) \\
h(k, k-L+2) \\
\vdots \\
h(k, k)
\end{bmatrix}
= \begin{bmatrix}
K(0) \\
K(1) \\
K(2) \\
\vdots \\
K(L)
\end{bmatrix}
$$

(15)
By the matrix inverse of the \(L+1\) square matrix Toeplitz matrix, the components of the finite impulse response function \(h(k,i) : k-L \leq i \leq k\), are obtained. The fixed-lag smoothing estimate \(\hat{z}(k-L,k)\) is calculated by (4) by the convolution of \(h(k,i)\) and the observed values \(y(i) \), \(k-L \leq i \leq k\).

Now, let us summarize the above results in Theorem 1.

**Theorem 1** Let the scalar observation equation be given by

\[
y(k) = z(k) + v(k).
\]

Here, \(v(k)\) represents the zero-mean white observation noise with the variance \(R\). Let the FIR smoothing estimate \(\hat{z}(k-L,k)\) of the signal \(z(k-L)\) be given by (4) as a linear convolution of the impulse response function \(h(k,i)\) and the observed values \(y(i)\), \(k-L \leq i \leq k\). From (15), the LS FIR impulse response function data are calculated by

\[
\begin{bmatrix}
h(k,k-L) \\
h(k,k-L+1) \\
h(k,k-L+2) \\
\vdots \\
h(k,k)
\end{bmatrix}
= \begin{bmatrix}
K_y(0) & K_y(1) & \cdots & K_y(L-1) & K_y(L) \\
K_y(1) & K_y(0) & \cdots & K_y(L-2) & K_y(L-1) \\
K_y(2) & K_y(1) & \cdots & K_y(L-3) & K_y(L-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_y(L) & K_y(L-1) & \cdots & K_y(1) & K_y(0)
\end{bmatrix}^{-1}
\begin{bmatrix}
K(0) \\
K(1) \\
K(2) \\
\vdots \\
K(L)
\end{bmatrix}.
\]

In (16), the necessary quantities in calculating the LS FIR smoothing estimate \(\hat{z}(k-L,k)\) are the auto-covariance function data of the signal process, \(K(i)\), \(1 \leq i \leq L\), the variance of the signal process, \(K(0)\), and the variance of the observation noise process, \(R\).

Let the FIR filtering estimate \(\hat{z}(k,k)\) of the signal \(z(k)\) be given by (5) as a linear convolution of the impulse response function \(h_f(k,i)\) and the observed values \(y(i)\), \(k-L \leq i \leq k\). From (10), the LS finite impulse response function data are calculated by

\[
\begin{bmatrix}
h_f(k,k-L) \\
h_f(k,k-L+1) \\
h_f(k,k-L+2) \\
\vdots \\
h_f(k,k)
\end{bmatrix}
= \begin{bmatrix}
K_y(0) & K_y(1) & \cdots & K_y(L-1) & K_y(L) \\
K_y(1) & K_y(0) & \cdots & K_y(L-2) & K_y(L-1) \\
K_y(2) & K_y(1) & \cdots & K_y(L-3) & K_y(L-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_y(L) & K_y(L-1) & \cdots & K_y(1) & K_y(0)
\end{bmatrix}^{-1}
\begin{bmatrix}
K(L) \\
K(L-1) \\
K(L-2) \\
\vdots \\
K(0)
\end{bmatrix}.
\]

In comparison of (17) with (16), there exists the following relationship between the LS FIR impulse response functions’ data in the FIR smoother and those in the FIR filter.

\[
\begin{bmatrix}
h_f(k,k-L) \\
h_f(k,k-L+1) \\
h_f(k,k-L+2) \\
\vdots \\
h_f(k,k)
\end{bmatrix}
= \begin{bmatrix}
h(k,k) \\
h(k,k-1) \\
h(k,k-2) \\
\vdots \\
h(k,k-L)
\end{bmatrix}.
\]

As an alternative computation method of the matrix inversion in Theorem 1, Theorem 2 proposes the Levinson-Durbin algorithm, which reduces the arithmetic operations of the order of \(O((L+1)^3)\) by the Gauss-Jordan elimination.
method to the order $O\left((L+1)^2\right)$ by the Levinson-Durbin algorithm.

**Theorem 2** By noting that $K(i) = K_\gamma(i), 1 \leq i \leq L$, and $K(0) + R = K_\gamma(0)$, in the Toeplitz matrix of (16), the finite impulse response function data $h(k, k-i) = h^{(L)}(i), 0 \leq i \leq L$, satisfy

$$
\begin{bmatrix}
K_\gamma(0) & K_\gamma(1) & \ldots & K_\gamma(L-1) & K_\gamma(L) \\
K_\gamma(1) & K_\gamma(0) & \ldots & K_\gamma(L-2) & K_\gamma(L-1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_\gamma(L) & K_\gamma(L-1) & \ldots & K_\gamma(1) & K_\gamma(0)
\end{bmatrix}
\begin{bmatrix}
h^{(L)}(L) \\
h^{(L)}(L-1) \\
\vdots \\
h^{(L)}(0)
\end{bmatrix}
= \begin{bmatrix}
K(0) \\
K(1) \\
\vdots \\
K(L)
\end{bmatrix}
$$

(19)

in estimating the signal at the start time of the FIR smoother.

Let $K^{(L)}$ be given by $K^{(L)} = \begin{bmatrix}
K_\gamma(0) & K_\gamma(1) & \ldots & K_\gamma(L-1) & K_\gamma(L) \\
K_\gamma(1) & K_\gamma(0) & \ldots & K_\gamma(L-2) & K_\gamma(L-1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_\gamma(L) & K_\gamma(L-1) & \ldots & K_\gamma(1) & K_\gamma(0)
\end{bmatrix}$, let $\tilde{h}^{(L)}$ be given by

$$
\tilde{h}^{(L)} = \begin{bmatrix}
h^{(L)}(L) \\
\vdots \\
h^{(L)}(0)
\end{bmatrix}
$$

and let $\tilde{k}^{(L)}_{cc}$ be given by $\tilde{k}^{(L)}_{cc} = \begin{bmatrix}
K(0) \\
K(1) \\
\vdots \\
K(L)
\end{bmatrix}$, then (19) is expressed as

$$
K^{(L)}\tilde{h}^{(L)} = \tilde{k}^{(L)}_{cc}
$$

(20)

Let $\tilde{h}^{(L)}$ be given by $\tilde{h}^{(L)} = \begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(L)
\end{bmatrix}$, let $\tilde{k}^{(L)}_{yy}$ be given by $\tilde{k}^{(L)}_{yy} = \begin{bmatrix}
K_\gamma(L) \\
\vdots \\
K_\gamma(1)
\end{bmatrix}$.

then the impulse response function, as the components of the vector $\tilde{h}^{(L)}$, is calculated by the Levinson-Durbin algorithm.

$$
\begin{align}
\tilde{h}^{(L)}(0) &= \frac{K(L) - (\tilde{k}^{(L)}_{yy})^T \tilde{h}^{(L-1)}}{K_\gamma(0) - (\tilde{k}^{(L)}_{yy})^T \tilde{h}^{(L)}} , \\
\tilde{h}^{(L)}(0) &= \frac{K(1)}{K_\gamma(0)} = \frac{K(1)}{K(0) + R} ,
\end{align}
$$

(21)

Here, $\tilde{h}^{(L)}$ satisfies

$$
K^{(L)}\tilde{h}^{(L)} = \tilde{k}^{(L)}_{yy}.
$$

(22)
Let $\tilde{h}^{(L)}$ be given by $\tilde{h}^{(L)} = \begin{bmatrix} \tilde{h}^{(L)}(1) \\ \vdots \\ \tilde{h}^{(L)}(L) \end{bmatrix}$, then $\tilde{h}^{(L)}$ is calculated by the Levinson-Durbin algorithm.

$$
\tilde{h}^{(L)}(L) = \frac{K_y(L) - (\tilde{h}^{(L-1)}(L))\tilde{h}^{(L-1)}}{K_y(0) - (\tilde{h}^{(L-1)}(L))\tilde{h}^{(L-1)}}, \quad \tilde{h}^{(1)}(0) = \frac{K_y(1)}{K_y(0)} = \frac{K_y(1)}{K(0) + R}.
$$

(23)

From (18), the LS finite impulse response data $h_f(k, k-i) = \tilde{h}^{(L)}(i)$, $0 \leq i \leq L$, for the FIR filtering estimate are given by

$$
\begin{bmatrix}
    h_f^{(L)}(L) \\
    h_f^{(L)}(L-1) \\
    h_f^{(L)}(L-2) \\
    \vdots \\
    h_f^{(L)}(0)
\end{bmatrix}
= \begin{bmatrix}
    h^{(L)}(0) \\
    h^{(L)}(1) \\
    h^{(L)}(2) \\
    \vdots \\
    h^{(L)}(L)
\end{bmatrix}.
$$

(24)

Proof of Theorem 2 is deferred to the appendix.

In section 4, the LS FIR smoothing error variance function of the signal process is developed from the viewpoint of the stability of the proposed LS FIR smoothing algorithm.

4. **LS FIR Smoothing Error Variance Function**

The variance function $P_z(k-L, k)$ of the FIR smoothing error $\tilde{z}(k-L) - \hat{z}(k-L, k)$ is expressed as follows.

$$
P_z(k-L, k) = E[(\tilde{z}(k-L) - \hat{z}(k-L, k))^2]
$$

$$
= K(k-L, k-L) - E[\hat{z}(k-L, k)^2]
$$

$$
= K(k-L, k-L) - \sum_{i=k-L}^{k} h(k, i) E[z(i)z(k-L)] = K(k-L, k-L) - \sum_{i=k-L}^{k} h(k, i) K(i, k-L)
$$

(25)

Here, we used the expression (3) for the auto-covariance function of the signal process. From (25) it is seen that the variance function $P_z(k-L, k) = E[\hat{z}(k-L, k)^2]$ of the FIR smoothing estimate $\hat{z}(k-L, k)$ is lower bounded by 0 and upper bounded by the variance function $K(k-L, k-L) = K(0)$ of the signal process. Namely, the inequality

$$
0 \leq P_z(k-L, k) \leq K(0)
$$

(26)

holds. This indicates that the proposed LS FIR smoothing algorithm is stable.

5. **A Numerical Simulation Example**

Next, let us estimate the signal which is obtained by multiplying the amplitude of the original sound data of the ‘laughter’ wav sound by 2. The sampling frequency of the sound signal process is 8.172 kHz. In the simulation, the auto-covariance function data of the signal process are obtained by use of the 350 sampled signal data. Substituting the auto-
covariance function data $K(i)$, $1 \leq i \leq L$, of the signal process, the variance $K(0)$ of the signal process and the variance $R$ of the observation noise process into (15), we can calculate the LS finite impulse response function $h(k,i)=h(k-i), k-L \leq i \leq k$. From (4), we can calculate the LS FIR smoothing estimate $\hat{z}(k-L,k)$ estimating the signal $z(k-L)$ as the convolution of the impulse response function data $h(k-i)$ and the observed values $y(i), k-L \leq i \leq k$. Figure 1 illustrates the signal $z(k)$ and the LS FIR smoothing estimate $\hat{z}(k,k+L)$ vs. $k, 1 \leq k \leq 450$, for the signal to noise ratio (SNR) 5 [dB] and the fixed interval $L=50$. Figure 2 illustrates the MSVs of the FIR filtering and smoothing errors by the proposed LS FIR filter and smoother in Theorem 1 vs. the fixed interval $L$, $10 \leq L \leq 100$, for SNR = 5, 10, 20 [dB]. In Figure 2, the estimation accuracy of the LS FIR smoother is almost same as the LS FIR filter for SNR=5, 10, 20 [dB]. For SNR=5, 10 [dB], as the fixed interval $L$ increases for $10 \leq L \leq 50$, there is a tendency that the MSVs of the FIR smoothing and filtering errors decrease. For SNR=20 [dB], as the fixed interval $L$ increases for $10 \leq L \leq 100$, the MSVs of the FIR filtering and smoothing errors show almost the flat characteristics. Figure 3 shows the MSVs of the FIR filtering and smoothing errors by the Levinson-Durbin algorithm in Theorem 2 vs. $L$, $10 \leq L \leq 100$, for SNR = 5, 10, 20 [dB]. From Figure 3, it is shown that the MSVs of the FIR filtering and smoothing errors are almost same for SNR = 5, 10, 20 [dB]. From Figure 2 and Figure 3, for SNR=5, 10 [dB], the estimation accuracies of the FIR filter and smoother by the Levinson-Durbin algorithm are inferior to the FIR filter and the smoother in Theorem 1. For SNR=20 [dB], the estimation accuracies of the FIR filter and smoother by the Levinson-Durbin algorithm are almost same as the FIR filter and the smoother in Theorem 1 respectively.

Here, the MSVs are calculated by $\sum_{k=1}^{500} (z(k)-\hat{z}(k,k))^2 / 500$ for the LS FIR filter and $\sum_{k=1}^{500} (z(k)-\hat{z}(k,k+L))^2 / 500$, for the LS FIR smoother.

Figure 1. Signal $z(k)$ and the LS FIR smoothing estimate $\hat{z}(k,k+L)$ vs. $k, 1 \leq k \leq 450$, for the SNR=5 [dB] and the fixed interval $L=50$. 
Figure 2. MSVs of the filtering and smoothing errors by the proposed LS FIR filter and smoother in Theorem 1 vs. \( L, 10 \leq L \leq 100, \) for \( \text{SNR} = 5, 10, 20 \) [dB].

Figure 3. MSVs of the filtering and smoothing errors by the Levinson-Durbin algorithm in Theorem 2 vs. \( L, \)
\( 10 \leq L \leq 100, \) for \( \text{SNR} = 5, 10, 20 \) [dB].
6. Conclusions

In this paper, the LS FIR smoothing and filtering algorithms are proposed in Theorem 1. The smoothing and filtering algorithms use the auto-covariance function data of the signal process, the variances of the signal process and the observation noise process.

In Theorem 2, the Levinson-Durbin algorithm for the finite impulse response function data is proposed in the FIR smoothing problem. The Levinson-Durbin algorithm needs less arithmetic operations than the Gauss-Jordan elimination method in the matrix inversion.

The numerical simulation results show that, by the Toeplitz matrix inversion method in Theorem 1, the estimation accuracy of the LS FIR smoother is almost same as the LS FIR filter for SNR=5, 10, 20 [dB]. For SNR=20 [dB], as the fixed interval \( L \) increases for \( 10 \leq L \leq 50 \), there is a tendency that the MSVs of the FIR filtering and smoothing errors decrease. For SNR=20 [dB], as the fixed interval \( L \) increases for \( 10 \leq L \leq 100 \), the MSVs of the FIR filtering and smoothing errors show almost the flat characteristics. It is also shown that the MSVs of the FIR filtering and smoothing errors by the Levinson-Durbin algorithm are almost same for the respective SNR. For SNR=20 [dB], the estimation accuracies of the FIR filter and smoother by the Levinson-Durbin algorithm are inferior to the FIR filter and smoother in Theorem 1. This result might have been caused by using the auto-covariance and variance information of the observation process instead of the auto-covariance and variance information of the signal process in the Toeplitz matrix inversion.

Also, in section 4, it is shown that the proposed LS FIR smoother is stable.

Appendix

Expanding (19), we have the following equations

\[
K_y(0) h^{(L)}(L) + K_y(1) h^{(L)}(L-1) + \cdots + K_y(L-1) h^{(L)}(1) + K_y(L) h^{(L)}(0) = K(0),
\]

\[
\vdots
\]

\[
K_y(L-1) h^{(L)}(L) + K_y(L-2) h^{(L)}(L-1) + \cdots + K_y(0) h^{(L)}(1) + K_y(1) h^{(L)}(0) = K(L-1),
\]

\[
K_y(L) h^{(L)}(L) + K_y(L-1) h^{(L)}(L-1) + \cdots + K_y(1) h^{(L)}(1) + K_y(0) h^{(L)}(0) = K(L).
\]

The first \( L \) equations in (A-1) are expressed as follows.

\[
\begin{bmatrix}
K_y(0) & \cdots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \cdots & K_y(0)
\end{bmatrix} \begin{bmatrix}
h^{(L)}(L) \\
h^{(L)}(L-1) \\
\vdots \\
h^{(L)}(1)
\end{bmatrix} = \begin{bmatrix}
K(0) \\
K(L-1) \\
\vdots \\
K_y(1)
\end{bmatrix} - h^{(L)}(0) 
\]

(A-2)

From (A-2), we have

\[
\begin{bmatrix}
h^{(L)}(L) \\
h^{(L)}(L-1) \\
\vdots \\
h^{(L)}(1)
\end{bmatrix} = \begin{bmatrix}
K_y(0) & \cdots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \cdots & K_y(0)
\end{bmatrix}^{-1} \begin{bmatrix}
K(0) \\
K(L-1) \\
\vdots \\
K_y(1)
\end{bmatrix} 
\]

(A-3)

\[
-h^{(L)}(0) = \begin{bmatrix}
K_y(0) & \cdots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \cdots & K_y(0)
\end{bmatrix}^{-1} \begin{bmatrix}
K_y(L) \\
K_y(1)
\end{bmatrix}.
\]

Now, from (A-1), in the case of \( L \) simultaneous equations, we find that
\[
\begin{bmatrix}
K_y(0) & \cdots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \cdots & K_y(0)
\end{bmatrix}
\begin{bmatrix}
h^{(L-1)}(L) \\
\vdots \\
h^{(L-1)}(1)
\end{bmatrix}
= 
\begin{bmatrix}
K(0) \\
\vdots \\
K(L-1)
\end{bmatrix}
\]  
(A-4)

(A-1) is expressed in matrix form as
\[
K_y^{(L)} \vec{h}^{(L)} = k_y^{(L)}.
\]  
(A-5)

Also, introducing the vector
\[
\vec{h}^{(L)} = 
\begin{bmatrix}
h^{(L)}(1) \\
\vdots \\
h^{(L)}(L)
\end{bmatrix},
\]  
(A-6)

which satisfies
\[
\begin{bmatrix}
K_y(0) & \cdots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \cdots & K_y(0)
\end{bmatrix}
\begin{bmatrix}
h^{(L)}(1) \\
\vdots \\
h^{(L)}(L)
\end{bmatrix}
= 
\begin{bmatrix}
K_y(L) \\
\vdots \\
K_y(1)
\end{bmatrix},
\]  
(A-7)

we obtain
\[
\vec{h}^{(L)} = \vec{h}^{(L-1)} - h^{(L)}(0)\vec{h}^{(L)}.
\]  
(A-8)

From the last equation in (A-1), we have
\[
\begin{bmatrix}
K_y(L) & K_y(L-1) & \cdots & K_y(1)
\end{bmatrix}
\begin{bmatrix}
h^{(L)}(L) \\
\vdots \\
h^{(L)}(1)
\end{bmatrix}
= K(L) - K_y(0)h^{(L)}(0).
\]  
(A-9)

Introducing
\[
\vec{k}^{(L)}_{yy} = 
\begin{bmatrix}
K_y(L) \\
\vdots \\
K_y(1)
\end{bmatrix},
\]  
(A-10)

we can rewrite (A-9) as
\[
\left(\vec{k}^{(L)}_{yy}\right)^T \vec{h}^{(L)} = K(L) - K_y(0)h^{(L)}(0).
\]  
(A-11)

Substituting (A-8) into (A-11) and arranging the equation, we obtain
\[
\vec{h}^{(L)}(0) = \frac{K(L) - \left(\vec{k}^{(L)}_{yy}\right)^T \vec{h}^{(L-1)}}{K_y(0) - \left(\vec{k}^{(L)}_{yy}\right)^T \vec{h}^{(L)}}.
\]  
(A-12)

Here, the initial condition \(\vec{h}^{(L)}(0)\) obtained from (A-1).
Reversing the order of the simultaneous equations of (A-7), we have
\[
\begin{bmatrix}
K_y(0) & \ldots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \ldots & K_y(0)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(L)
\end{bmatrix}
= \begin{bmatrix}
K_y(1) \\
\vdots \\
K_y(L)
\end{bmatrix}.
\] (A-13)

Introducing the vector function \( \tilde{h}^{(L)} \) given by
\[
\tilde{h}^{(L)} = \begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(L)
\end{bmatrix}
= \begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(1)
\end{bmatrix},
\] (A-14)
and \( \tilde{k}^{(L)}_{xy} \) given by
\[
\tilde{k}^{(L)}_{xy} = \begin{bmatrix}
K_y(1) \\
\vdots \\
K_y(L)
\end{bmatrix},
\] (A-15)
we express (A-13) in matrix form as
\[
K^{(L)}\tilde{h}^{(L)} = \tilde{k}^{(L)}_{xy}.
\] (A-16)

Let us express (A-16) in terms of the matrix elements and the vector components as
\[
\begin{bmatrix}
K_y(0) & \ldots & K_y(L-1) \\
\vdots & \ddots & \vdots \\
K_y(L-1) & \ldots & K_y(0)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(L)
\end{bmatrix}
= \begin{bmatrix}
K_y(1) \\
\vdots \\
K_y(L)
\end{bmatrix}.
\] (A-17)

Expanding (A-17), we have the following equations
\[
K_y(0)\tilde{h}^{(L)}(1) + K_y(1)\tilde{h}^{(L)}(2) + \cdots + K_y(L-2)\tilde{h}^{(L)}(L-1) + K_y(L-1)\tilde{h}^{(L)}(L) = K_y(1),
\]
\[
\vdots
\]
\[
K_y(L-2)\tilde{h}^{(L)}(1) + K_y(L-3)\tilde{h}^{(L)}(2) + \cdots + K_y(0)\tilde{h}^{(L)}(L-1) + K_y(1)\tilde{h}^{(L)}(L)
= K_y(L-1),
\]
\[
K_y(L-1)\tilde{h}^{(L)}(1) + K_y(L-2)\tilde{h}^{(L)}(2) + \cdots + K_y(1)\tilde{h}^{(L)}(L-1) + K_y(0)\tilde{h}^{(L)}(L) = K_y(L).
\] (A-18)

The first \( L-1 \) equations in (A-18) are expressed as follows.
\[
\begin{bmatrix}
K_y(0) & \ldots & K_y(L-2) \\
\vdots & \ddots & \vdots \\
K_y(L-2) & \ldots & K_y(0)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(L-1)
\end{bmatrix}
= \begin{bmatrix}
K_y(1) \\
\vdots \\
K_y(L-1)
\end{bmatrix} - \begin{bmatrix}
\tilde{h}^{(L)}(L) \\
\vdots \\
\tilde{h}^{(L)}(L-1)
\end{bmatrix}.
\] (A-19)

From (A-19), we have
\[
\begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\vdots \\
\tilde{h}^{(L)}(L-1)
\end{bmatrix} =
\begin{bmatrix}
K_y(0) & \ldots & K_y(L-2) \\
\vdots & \ddots & \vdots \\
K_y(L-2) & \ldots & K_y(0)
\end{bmatrix}^{-1}
\begin{bmatrix}
K_y(1) \\
\vdots \\
K_y(L-1)
\end{bmatrix}.
\] (A-20)

Now, with relation to (A-18), in the case of \( L-1 \) simultaneous equations,
\[
\begin{bmatrix}
K_y(0) & \ldots & K_y(L-2) \\
\vdots & \ddots & \vdots \\
K_y(L-2) & \ldots & K_y(0)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^{(L-1)}(1) \\
\vdots \\
\tilde{h}^{(L-1)}(L-1)
\end{bmatrix} =
\begin{bmatrix}
K_y(1) \\
\vdots \\
K_y(L-1)
\end{bmatrix}.
\] (A-21)

we can express (A-21) in matrix form as
\[
K^{(L-1)} \tilde{h}^{(L-1)} = \tilde{k}^{(L-1)}.
\] (A-22)

From (A-6), (A-7), (A-16) and (A-20), we obtain
\[
\tilde{h}^{(L)} = \tilde{h}^{(L-1)} - \tilde{h}^{(L)}(L)\tilde{k}^{(L-1)}.
\] (A-23)

From the last equation in (A-18), we have
\[
\begin{bmatrix}
K_y(L-1) & K_y(L-2) & \ldots & K_y(1)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^{(L)}(1) \\
\tilde{h}^{(L)}(2) \\
\vdots \\
\tilde{h}^{(L)}(L-1)
\end{bmatrix} = K_y(L) - K_y(0)\tilde{h}^{(L)}(L).
\] (A-24)

From (A-10), we can rewrite (A-24) as
\[
\left(\tilde{k}^{(L-1)}\right)^T \tilde{h}^{(L-1)} = K_y(L) - K_y(0)\tilde{h}^{(L)}(L).
\] (A-25)

Substituting (A-23) into (A-25) and arranging the equation, we obtain
\[
\begin{align*}
\tilde{h}^{(L)}(L) &= \frac{K_y(L) - (\tilde{k}^{(L-1)})^T \tilde{h}^{(L-1)}}{K_y(0) - (\tilde{k}^{(L-1)})^T \tilde{h}^{(L-1)}} \\
K_y(0) &= \frac{K_y(1)}{K_y(0)} = \frac{K_y(1)}{K(0) + R}.
\end{align*}
\] (A-26)

Here, the initial condition \( \tilde{h}^{(L)}(0) \) in (A-26) is obtained from (A-17).

(Q.E.D.)

References


**Author Biography**

**Seiichi Nakamori** is a Specially Appointed Professor in the Department of Technology, Faculty of Education, Kagoshima University, from April, 2017 to date. He was appointed Professor Emeritus, Kagoshima University, in April 2016. He was a Professor, Department of Technology, Graduate School of Education, Kagoshima University from April, 1994 to March 2016, Associate Professor, in the Department from April, 1987 to March, 1994 and a lecturer, Interdisciplinary Chair (Applied Mathematics), Faculty of Engineering, Oita University, from April, 1985 to March, 1987. He received the B.E. degree in Electronic Engineering from Kagoshima University in 1974 and the Dr. Eng. Degree in Applied Mathematics and Physics from Kyoto University in 1982. He is mainly interested in stochastic signal estimation, image restoration and constant control of temperature by using microcontroller.