Mathematical Modeling of Wave Fields in Layered Media with Additional Stresses

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Abstract

A horizontally layered elastic structure containing homogeneous solid layers is considered. The $i^{th}$ layer of this structure is located under uniaxial compression / tension. The paper is aimed on studying interaction of elastic waves, caused by local pulse source, with the structure. The boundary-value problem, describing the wave dynamics of the structure, is formulated. The additional stress causes the anisotropy of parameters of the medium. The matrix method is used to calculate the wavefield on the free surface from source of acoustic waves presented by moment tensor. The impact of hydrostatic pressure on wave interference pattern on free surface is analysed. It was established, that wave interference pattern on the free surface of the structure is dependent on value of additional uniaxial compression / tension of $i^{th}$ layer. Obtained results can be used to develop methods for evaluation the impact of velocity anisotropy in the medium.


Introduction

The static stress change the propagation velocity of elastic disturbances in solids and therefore may influence on wave interference pattern on free surface. Despite, that the changes of the elastic properties and density of solids under stress is quite small they can significantly affect the interference patterns fields of elastic disturbances. Under certain conditions this phenomenon could be used for remote determination of stresses acting in layers of geological structures. The quantification of the impact of the initial stress in geological structures on the interference wave pattern is essential. In the paper the influence of hydrostatic pressure acting in one of the layer of horizontally-layered structure on the wave field, is investigated. The source of seismic waves is point in the space, but distributed in time.

The analytic ratios for displacement field by matrix method Thomson-Haskell, is obtained. We modeled the layered medium as pack of homogeneous isotropic layers with parallel boundaries, except those layers that are under additional stress. The source of seismic waves is presented by moment tensor. On the boundaries between the layers the conditions of hard contact are implementing. The free surface is free from stress. The source of seismic waves is located on a defined depth $x_3 = x_3^0$. The waves from half-space (n+1) are not returned (radiation condition). The $i^{th}$ layer is under the influence of additional stress as hydrostatic pressure.

Problem Formulation

We consider the semi-infinite layered structure, consisting of two infinite elastic layers $L_1$, $L_2$ and elastic half-space $L_3$ (fig. 1):

$L_1 = \{ -\infty < x_1 < \infty, -\infty < x_2 < \infty, 0 < x_3 < h_1 \}$,
$L_2 = \{ -\infty < x_1 < \infty, -\infty < x_2 < \infty, h_1 < x_3 < h_1 + h_2 \}$,
$L_3 = \{ -\infty < x_1 < \infty, -\infty < x_2 < \infty, h_1 + h_2 < x_3 < \infty \}$,

Here $x_i$, $i = 1, 2, 3$ stand for Cartesian coordinates, $h_i$ are the thickness of each layer.
Wave field in each layer satisfies three groups of relations of the theory of elasticity [1]:

Equations of motion

\[
\rho_\alpha \frac{\partial^2 u_{\alpha j}^i}{\partial t^2} = \frac{\partial \sigma_{\alpha j}^i}{\partial x_i} + \rho_\alpha f_{\alpha j}^i, \quad j = 1, 2, 3, \quad (1)
\]

elasticity relations

\[
\sigma_{\alpha j}^i = 2 \mu_\alpha e_{\alpha j}^i + \lambda_\alpha e_{\alpha k}^i \delta_{k j} \delta_{\alpha}, \quad (2)
\]

and compatibility relations (in Cauchy form)

\[
e_{\alpha j}^i = \frac{1}{2} \left( \frac{\partial u_{\alpha j}^i}{\partial x_j} + \frac{\partial u_{\alpha j}^i}{\partial x_i} \right). \quad (3)
\]

Here \( \rho_\alpha \) is mass density of elastic medium in \( \alpha \)-layer (\( \alpha = 1, 2, 3 \)), \( u_{\alpha j}^i \), \( \sigma_{\alpha j}^i \), \( e_{\alpha j}^i \) stand for Cartesian components of displacement vector, stress tensor, and strain tensor correspondingly of \( \alpha \)-layer. \( \lambda_\alpha \) and \( \mu_\alpha \) are elastic moduli (Lame coefficients) \( f_{\alpha j}^i = f_{\alpha j}^i \left( x_1, x_2, x_3, t \right) \) stand for \( i-th \) Cartesian component of body force in \( \alpha \)-layer.

With the use of the relations (2) and (3) the equation (1) can be reduced to the hyperbolic system of equations [8]

\[
\rho_\alpha \frac{\partial^2 u_{\alpha j}^i}{\partial t^2} = \mu_\alpha \frac{\partial^2 u_{\alpha j}^i}{\partial x_i \partial x_j} \delta_{k j} + (\lambda_\alpha + \mu_\alpha) \frac{\partial}{\partial x_i} \left( \frac{\partial u_{\alpha i}^k}{\partial x_j} \delta_{k j} \right) + f_{\alpha j}^i. \quad (4)
\]

The surface \( x_3 = 0 \) is free of traction

\[
\sigma_{31}^i \big|_{x_3=0} = 0, \quad \sigma_{32}^i \big|_{x_3=0} = 0, \quad \sigma_{33}^i \big|_{x_3=0} = 0. \quad (5)
\]

We consider the interfaces between the layers as the boundary of ideal elastic contact, what can written as

\[
u_{\alpha j}^i \bigg|_{x_3=h_3} = u_{\beta j}^i \bigg|_{x_3=h_3},
\]

\[
\sigma_{\alpha j}^i \bigg|_{x_3=h_3} = \sigma_{\beta j}^i \bigg|_{x_3=h_3},
\]

\[
u_{\alpha j}^i \bigg|_{x_3=h_3+h_2} = u_{\beta j}^i \bigg|_{x_3=h_3+h_2},
\]

\[
\sigma_{\alpha j}^i \bigg|_{x_3=h_3+h_2} = \sigma_{\beta j}^i \bigg|_{x_3=h_3+h_2}. \quad (7)
\]
According to the relations (2), (3) the stress components $\sigma_{3i}^{\alpha}$ in the relations (6) – (7) are considered and function of displacement components $u_i^{\alpha}$.

We suppose additionally that function $u_i^{\alpha}(x_1, x_2, x_3, t)$ satisfy Sommerfeld radiation condition [9] in infinitely remote points.

When the body forces $f_i^{\alpha}(x_1, x_2, x_3, t)$ are given as function of spatial coordinates and time, the problem (4) – (8) describes the wave field in the structure.

Let there is an only source, which is localized in the layer $L_i$ at the point $(x_1^s, x_2^s, x_3^s)$, where $x_3^s \in (0, h_i)$:

$$f_i^I = \varphi(t) \delta(x_1 - x_1^s) \delta(x_2 - x_2^s) \delta(x_3 - x_3^s), \quad f_i^{\alpha} = 0, \quad \alpha = II, III, \quad (8)$$

Here $\varphi(t)$ is a function determining dependence in time of the source.

Now we can replace the equations (1) by corresponding homogeneous equations in each layer. To do that we will consider the layer $L_\alpha$ as the union of two layers

$$(9)$$

The Method for Solving the Problem

Now the homogeneous equations of motion acts in each layer $\alpha = 1, 2, 3$ [2, 6]

$$\rho^\alpha \frac{\partial^2 u_i^{\alpha}}{\partial t^2} - \frac{\partial \sigma_{ij}^{\alpha}}{\partial x_j} = 0, \quad j = 1, 2, 3, \quad (10)$$

The elasticity moduli of layers $L_\alpha$ and $L_{\alpha'}$ are defined as

$$\lambda_\alpha = \lambda_{\alpha'}, \quad \mu_\alpha = \mu_{\alpha'} = \mu_\alpha.$$  

To take into account the source, acting in the interface between $L_\alpha$ and $L_{\alpha'}$, we present the contact conditions on the boundary $x_3 = x_{3i}^s$ in the form

$$u_i^I \bigg|_{x_3 = x_{3i}^s} - u_i^{II} \bigg|_{x_3 = x_{3i}^s} = u_i^I, \quad \sigma_{3i}^I \bigg|_{x_3 = x_{3i}^s} - \sigma_{3i}^{II} \bigg|_{x_3 = x_{3i}^s} = \sigma_{3i}^I. \quad (11)$$

Now instead of relations (6) we have

$$u_i^I \bigg|_{x_3 = L-H} = u_i^{II} \bigg|_{x_3 = L-H}, \quad \sigma_{3i}^I \bigg|_{x_3 = L-H} = \sigma_{3i}^{II} \bigg|_{x_3 = L-H}. \quad (12)$$

We will consider pulse sources $f_i^{\alpha}$ which varying in time like a peak shape function, for instance, like Gaussian $f(t) = f_0 \exp((t-t_0)^2/b^2)$ or Lorenzian $f(t) = b^2/(b^2 + (t-t_0)^2)$. This enables us to apply the Fourier transform to them.
Substituting relation (3) into (2), we obtain

\[
\mu^\alpha \left( \frac{\partial u_{ij}^\alpha}{\partial x_j} + \frac{\partial u_{ij}^\alpha}{\partial x_i} \right) + \lambda^\alpha \sum_{k=1}^{3} \frac{\partial u_{ik}^\alpha}{\partial x_j} - \sigma_{ij}^\alpha = 0. \tag{13}
\]

So, we have the system (10), (13) of partial differential equations with respect to three component \( u_{ij}^\alpha \) of displacement vector and 6 components of stress tensor \( \{ \sigma_{ij}^\alpha \} \) (with accounting of its symmetry).

Under taken assumptions about behavior of the functions \( u_{ij}^\alpha (x_1, x_2, x_3, t) \), \( \sigma_{ij}^\alpha (x_1, x_2, x_3, t) \) and at infinity when \( x_1 \to \pm \infty \) and \( x_2 \to \pm \infty \) we can apply to them Fourier transform with respect special coordinate \( x_1 \) and \( x_2 \). The structure (9) of the source enables to use this integral transform to the functions \( u_{ij}^\alpha (x_1, x_2, x_3, t) \) and \( \sigma_{ij}^\alpha (x_1, x_2, x_3, t) \). And finely the assumption regarding the function \( f(t) \) enables apply the Fourier transform with respect to time to all these functions.

Applying to equations (10) and (13) integral Fourier transform with respect to variables \( x_1, x_2 \) and \( t \), we can reduce them to the system of 6 ordinary differential equations in independent variable \( x_3 \) with respect to variables \( U_{ij}^\alpha, U_{ij}^\alpha, U_{ij}^\alpha, S_1^\alpha, S_2^\alpha, S_3^\alpha \). The dependent variables are Fourier transforms of functions \( u_{ij}^\alpha, u_{ij}^\alpha, u_{ij}^\alpha, \sigma_{ij}^\alpha, \sigma_{ij}^\alpha, \sigma_{ij}^\alpha \) correspondingly:

\[
U_{ij}^\alpha \equiv \mathbf{F}(u_{ij}^\alpha), \quad S_{ij}^\alpha \equiv \mathbf{F}(\sigma_{ij}^\alpha), \tag{14}
\]

Here

\[
\mathbf{F}(\ldots) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -j \omega (t - x_1 p_1 - x_2 p_2) \right) dx_1 dx_2 dt,
\tag{15}
\]

where \( j \equiv \sqrt{-1}, \omega \) and \( p_1, p_2 \) are parameters of Fourier transforms with respect to time \( t \) and special variables \( x_1, x_2 \) (circular frequency and slownesses).

We can write the system in matrix form [3]

\[
\frac{d\mathbf{B}^\alpha}{dx_3} = j \omega \mathbf{A}^\alpha \cdot \mathbf{B}^\alpha. \tag{16}
\]

Here \( \mathbf{B}^\alpha \) stands for column-vector \( 6 \times 1 \) \( \mathbf{B}^\alpha = (U^\alpha, S^\alpha)^T \), where \( \mathbf{U}^\alpha = (U_1^\alpha, U_2^\alpha, U_3^\alpha)^T \), \( \mathbf{S}^\alpha = -(j \omega)^{-1} (S_1^\alpha, S_2^\alpha, S_3^\alpha)^T \), \( \mathbf{A}^\alpha \) is \( 6 \times 6 \)-matrix.
The general solution can be presented via

\[ A^* = \begin{pmatrix}
  q_1 p_1 & q_1 p_2 & 0 & \cdots & 0 \\
  q_2 p_1 & q_2 p_2 & \cdots & 0 \\
  0 & p_1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & p_1
\end{pmatrix},
\]

where

\[ q_1 = -2 \rho \left( \frac{\rho^3}{\lambda^2 + 2 \mu^2} - \rho \right)^{\frac{1}{2}}, \quad q_2 = \frac{-2 \rho^2 p}{\left( \frac{\rho^2}{\mu^2} - \rho \right)^{\frac{1}{2}}}, \quad q_3 = \frac{-2 \rho^2 p}{\left( \frac{\rho^2}{\mu^2} - \rho \right)^{\frac{1}{2}}},
\]

\[ p^2 = p_1^2 + p_2^2. \]

As (16) is a system of ordinary differential equations with constant coefficients, its general solution can be presented via matrix exponential [4]

\[ B^t_a = \exp \left( j \omega (x_3 - x_0) A_a \right) \cdot B^0_a, \]

where \( x_3 \in (0, L_a) \), \( B^0_a \) is the initial value i.e. \( B^0_a = B_a \mid_{t=0} \).

The matrix exponential can be easily calculated by reducing the matrix \( A_a \) to normal Jordan form [7]

\[ A_a = T_a \cdot J_a \cdot T_a^{-1}. \]

Here \( J_a \) is the normal Jordan form of the matrix \( A_a \), \( T_a \) is non-singular matrix. \( J_a \) is an upper triangular quasidiagonal matrix: \( J_a = \text{diag} \left( J^a_0, J^a_1, \ldots, J^a_K \right) \). Block \( J^a_k \) is the diagonal matrix built of distinct eigenvalues \( \lambda^a_1, \lambda^a_2, \ldots, \lambda^a_M \) of the matrix \( A_a \): \( J^a_k = \text{diag} \left( \lambda^a_1, \lambda^a_2, \ldots, \lambda^a_M \right) \), where \( M \in \mathbb{N} \) is the number of distinct eigenvalues (real and complex). Each block \( J^a_k, k = 1, 2, \ldots, K \) corresponds to a repeated eigenvalue \( \lambda^a_\ell \). If \( r_\ell \in \mathbb{N} \) is the number of multiplicity of the eigenvalue \( \lambda^a_\ell \), then \( J^a_k \) is an \( r_\ell \times r_\ell \)-matrix \( J^a_k = \text{diag} \left( \lambda^a_1, \ldots, \lambda^a_\ell \right) + \mathbf{H}^a_k \), where \( \mathbf{H}^a_k \) is \( r_\ell \times r_\ell \)-matrix which first upper diagonal contains ones, all other its entries are zeros.

If matrix \( A_a \) do not have repeated eigenvalues, then solution (19) can be presented in the form

\[ B^t_a = T_a \cdot \left( \text{diag} \left( \exp \left( j \omega (x_3 - x_0) \lambda^a_1 \right), \ldots, \exp \left( j \omega (x_3 - x_0) \lambda^a_M \right) \right) \right) \cdot T_a^{-1} \cdot B^0_a.
\]

where \( T_a \) is \( 6 \times 6 \)-matrix, the columns of which are eigenvectors of the matrix \( A_a \).

Applying to relations (11), (12), (7), (8) Fourier transform (15) we obtain

\[ B_1 \mid_{x_1=x_3} - B_1 \mid_{x_1=x_3} = F, \quad B_1 \mid_{x_1=h_1} - B_2 \mid_{x_1=h_1} = 0, \quad B_2 \mid_{x_1=h_2} - B_3 \mid_{x_1=h_2} = 0,
\]

where

\[ F = (U^T, S^T)^T, \quad U^T = (U^T_1, U^T_2, U^T_3)^T, \quad S^T = (S^T_1, S^T_2, S^T_3)^T, \quad U^T_1 = F \left( U^T_1 \right), \quad S^T_1 = F \left( S^T_1 \right).
\]
The components $F_{\beta}, \beta = 1, 2, \ldots, 6$ of the source vector $F$ can be expressed via the components $M_{\beta\gamma}$ of tensor of seismic moment $M$ [3]:

$$F_1 = -\frac{M_{11}}{c_{55}}, F_2 = -\frac{M_{22}}{c_{44}}, F_3 = -\frac{M_{33}}{c_{33}}, F_4 = p_1 \left(\frac{M_{11}}{c_{11}} - \frac{c_{13}}{c_{33}} M_{33}\right) + p_2 M_{12},$$

$$F_5 = p_1 M_{21} + p_2 \left(\frac{c_{23}}{c_{33}} M_{33}\right), F_6 = p_1 (M_{31} - M_{13}) + p_2 (M_{32} - M_{23})$$

(24)

where $c_{13}, c_{23}, c_{33}, c_{44}$ and $c_{55}$ are stiffness matrix coefficients. In the case of isotropic medium:

$$c_{13} = c_{23} = \lambda, \quad c_{33} = \lambda + 2\mu, \quad c_{44} = c_{55} = \mu.$$

Application of Fourier transform to relations (5) yields

$$S_{\gamma} = 0.$$  

(25)

To find $B_\alpha(x_3, p_1, p_2, \omega)\forall \alpha = I', I'', II', III'$ we should determine values of unknown vectors $B_\alpha^0$ to $B_\alpha^m$. Each vector has 6 scalar constants. But because of radiation condition at $x_3 \to \infty$ only three components of vector $B_\alpha^0$ are independent. So, the whole number of scalar unknowns is 27. Substituting (19) into relations (22), we obtain 24 scalar equations. Relation (25) gives three equations. So, totally we have 27 equations.

To find components $u_i^\alpha (x_3, x_1, x_2, t)$ of displacement vector in the layers $L_\alpha, \alpha = I', I'', II', \ldots, IV$ we apply to corresponding $U_i^\alpha (x_3, p_1, p_2, w)$ inverse Fourier transform

$$u_i^\alpha = \mathbf{F}^{-1}(U_i^\alpha), \quad \mathbf{F}^{-1}(\ldots) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega^2 e^{i\omega t - p_1 x_1 - p_2 x_2} dp_1 dp_2 d\omega.$$  

(26)

Further we use this method to study wave field interference of the free surface $x_3 = 0$.

**The Calculation Uniaxial Compression / Tension Influence of**

We applied the approach developed in the previous sections to study a wave interference pattern on free surface of the layered structure. The $i^{th}$ layer (see. Fig. 1) is located under uniaxial compression; its physical properties are not isotropic. This layer has a hexagonal symmetry (transversely-isotropic), in which the physical properties are changed in a direction $x_3$. To describe such medium the stiffness matrix $\mathbf{c}_i$ has five independent parameters, which are determined by relations:

$$c_{11} = \lambda + 2\mu - \frac{2\sigma(l\mu + m\lambda)}{\mu(3\lambda + 2\mu)}, \quad c_{12} = \lambda - \frac{\sigma(-2l\mu + (2m - n)(\lambda + \mu))}{\mu(3\lambda + 2\mu)},$$

$$c_{13} = \lambda - \frac{\sigma(4l\mu + 2m\lambda - n\lambda)}{2\mu(3\lambda + 2\mu)}, \quad c_{33} = \lambda + 2\mu + \frac{2\sigma(l\mu + 2m\lambda + 2m\mu)}{\mu(3\lambda + 2\mu)},$$

$$c_{44} = \mu + \frac{\sigma(4m\mu + n\lambda)}{4\mu(3\lambda + 2\mu)}, \quad c_{66} = \mu + \frac{\sigma(2m\mu - n\lambda - m\mu)}{2\mu(3\lambda + 2\mu)}.$$  

(27)

where $\sigma$ stands for stress (if $\sigma < 0$, the layer is under compression, in case of $\sigma > 0$, the layer is under stretching); $l, m, n$ are Murnaghan coefficients.

Table. 1 shows the parameters of an isotropic medium.
Table 1. The parameters of medium

<table>
<thead>
<tr>
<th>N₀</th>
<th>h, m</th>
<th>λ, GPa</th>
<th>μ, GPa</th>
<th>ρ, kg/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3000</td>
<td>194.2</td>
<td>20.18</td>
<td>2100</td>
</tr>
<tr>
<td>2</td>
<td>5000</td>
<td>216.3</td>
<td>25.03</td>
<td>2370</td>
</tr>
<tr>
<td>3</td>
<td>20000</td>
<td>297.3</td>
<td>32.84</td>
<td>2650</td>
</tr>
</tbody>
</table>

The additional parameters of the i-th layer have the meanings:

\[ l = -33.71 \text{ GPa}, \ m = -66.42 \text{ GPa}, \ n = -66.00 \text{ GPa}. \]

Consider the time distributed source of seismic waves, which is presented by moment tensor \( M_{ij} \), all components of which are zero except \( M_{xz} \) and \( M_{zx} \) are equal \( 1.74 \times 10^{14} \text{ J} \). The source time function is wavelet function [4]. The source of seismic waves is located in the first layer at a depth of 2000m.

The wavefields are calculated at the epicenter distance of 9014m for different values of uniaxial compression / tension \( \sigma \) (\( \sigma = 0 \) (isotropic case), \( \sigma = 0.01\mu, 0.02\mu, 0.03\mu, 0.04\mu \), and \( \sigma = -0.01\mu, -0.02\mu, -0.03\mu, -0.04\mu \)).

Consider three problems, when the first layer is under uniaxial compression, the second layer is under uniaxial compression and the half-space is under uniaxial compression.

In the first problem, the source of acoustic waves is located in anisotropic layer, (layer with additional compression), we can see the impact of velocity anisotropy on the wave interference pattern for different values of uniaxial compression / tension \( \sigma \). In this case, for all values \( \sigma \) the time of P-, S-wave first arrival and wave-form are different. Also, the influence of velocity anisotropy is significant impact on exchange effects (fig. 2). On fig 3. is shown dependence the time of P-wave first arrival on the different values of uniaxial compression / tension. This dependence is nonlinear and has exponential form.

![Wave interference pattern](image)

Fig. 2. The wave interference pattern on free surface calculated for the first problem
Fig. 3. The dependence time of P-wave first arrival on the different values of uniaxial compression / tension

In the second problem, the source is located in the first isotropic layer and the second layer is under uniaxial compression / tension, the influence of velocity anisotropy is non-linear and perceptible only on exchange effects (fig. 4, 5).

Fig. 4. The wave interference pattern on free surface calculated for the second problem
Fig. 5. The exchange effects on the wave interference pattern calculated for the second problem

In the third problem, when the half-space is anisotropic and two other layers have isotropic properties, the impact of velocity anisotropy is only noticeable on exchange effects, and is much smaller than for other cases (fig. 6, 7).

Fig. 6. The wave interference pattern on free surface calculated for the third problem
Fig. 7. The wave interference pattern on free surface calculated for the third problem

Conclusion

If the source of seismic waves is located in $i^{th}$ layer, the time of first arrival of direct P- and S-waves and their wave forms change depending on different values of additional uniaxial compression / tension. If the source of seismic waves is located under $i^{th}$, the time of first arrival of direct P- and S-waves and their wave forms are the same different values of hydrostatic pressure. The influence of velocity anisotropy, caused by hydrostatic pressure, is significant on exchange waves. After analyzing seismograms on fig. 4 and fig. 6, we can conclude that additional hydrostatic compression in the layer under the source of seismic waves do not affect the direct P- and S-waves, and has little effect on exchange effects. On fig. 5 and fig. 7 is shown, the exchange waves, that arrival after direct S-waves, have different wave forms and time of arrival for different values of hydrostatic pressure.

References


